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GRAPHS AND ANONYMOUS SOCIAL WELFARE FUNCTIONS*

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1. INTRODUCTION

The purpose of this paper is threefold: first, to present a graph-theoretic tool to analyze restricted domains of preferences; second, to characterize domains, via the graphic approach, which permit construction of anonymous social welfare functions; and third, to use this approach to show that construction of such a function is independent of the number of voters.

The problem can best be explained by the following two-person example: Let x, y, z and w be "alternatives" over which the 2 voters have preferences (which constitute an ordering for each voter). The pool of preferences from which the voters take their ballots (called the domain of preference) is given by Ω where:

$$\Omega = \{yzxw,\ zxyw,\ xwyz,\ ywzx,\ xyzw,\ zwxy,\ zxwy,\ xywz,\ yzwx\}$$

where, for example, if a voter's ballot is $yzxw$, it implies that he prefers y to z to x to w.

From the two voters' ballots p_1 and p_2, one wishes to form a social welfare function $f(p_1, p_2)$ (the group's preference) which satisfies the three conditions of Independence, Pareto and nondictatorship (which are formally defined in the next section). If Ω includes all the possible 24 orderings, then Arrow's well-known result guarantees that no such function exists. When only some of the orderings are included in Ω, however, this conclusion does not necessarily hold. We can then ask the following question: can we say, by looking at the domain Ω, whether such functions exist and if so, how many?

For the 2-person case, the paper (in the next two sections) describes a systematic way to proceed from Ω to form a graph in order to construct all possible such functions. For the example at hand, we show (in the third section) that a social welfare function exists and is unique.

The graph is defined as follows: the nodes are ordered pairs of alternatives and a directed branch connects one pair to another if any coalition which is decisive over the first pair under any SWF in the class under consideration can be shown to be decisive over the second as well. The existence of a nondictatorial SWF is equivalent to the existence of a set of pairs with no outgoing branches; such a set is called a *sink*.

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A sink thus is a set of pairs (which is a proper subset of all pairs) such that no branch connects a pair inside the set to a pair outside the set. There might be ingoing branches though, i.e., branches which connect pairs outside the set to pairs which belongs to the set. Since we let voters (or coalitions) be decisive over sinks (and only sinks), the existence of a sink defines and limits the power of an individual voter (or a coalition).

Social welfare functions satisfying Arrow's nondictatorship requirement can exhibit very unequal distributions of power. That requirement rules out a dictator but not, say, a coalition of two voters being decisive over all pairs with the rest of the voters being dummies, i.e., not being decisive over any pair in any voting situation.

Thus, when constructing a SWF, one might wish to rule out the existence of dummies. This requirement is investigated in Blair and Muller [1981]. Imposing this condition, called essentiality (Fishburn [1973]), still does not guarantee a "fair" rule as can be demonstrated by the four alternatives paradox of voting configuration $\Omega = \{(xwyz), (wyzx), (yzxw), (zxwy)\}$. Let one individual be decisive for all pairs except for (zx). Let all other voters have *veto power* over zx, i.e., each one of them can block the outcome z over x by voting the reverse — x over z. Thus each one is individually decisive over the pair (xz). A straightforward inspection reveals that this is a transitive SWF but indeed all voters except one have very limited power.

Thus if a more equitable distribution of power is desired we can try to construct an anonymous rule. Under an anonymous SWF, all voters have equal power. In the fourth section, I investigate the question of the existence of an anonymous rule. I show that existence is independent of the number of voters (as long as it is more than three). A *transitive sink* is defined as a sink whose pairs form a transitive ordering. Thus the set $\{xy, yz, xz\}$ forms the ordering (xyz) while $\{xy, yz, zx\}$ does not form an ordering. Theorem 3 shows that the existence of an anonymous SWF is equivalent to the existence of a transitive sink.

2. DOMAINS AND DERIVED GRAPHS[2]

Let A be a set of alternatives with at least two elements and let Σ be the set of all strong preferences on A. Let $\Omega \subset \Sigma$ be the admissible preferences. An n-

[2] Formally the domain used here is Ω^n where $\Omega \subset \Sigma$, while the more commonly used (see Sen and Pattanaik [1969]) is $\Omega \subset \Sigma^n$.

Those are two distinct approaches since the second is a restriction on *profiles* of preferences, while the first is on the "pool" of preferences from which each individual draws his vote. To see that better, one can look at the Extremal Restriction (ER) condition of Sen [1979, p. 174] which reads: "if for an ordered triple (xyz) there is someone who prefers x to y and y to z, then anyone regards z to be uniquely best if and only if he regards x to be uniquely worst". Thus to see whether majority rule cycles or not, one has to check the given profile of preferences each time there is a vote, and check whether in this vote if someone voted (xyz) no one voted (zxy) or (yzx).

(*Continued on next page*)

person *social welfare function* on Ω is a function $f: \Omega^n \to \Sigma$ which satisfies the *Pareto criterion* (xp_iy for all i implies $xf(p_1,\ldots, p_n)y$) and *Independence of Irrelevant Alternatives*: if [xp_iy if and only if xq_iy for all i] then $xf(p_1,\ldots, p_n)y$ if and only if $xf[q_1,\ldots, q_n)y]$. f is *nondictatorial* if there does not exist an i for which $f(p_1,\ldots, p_n)=p_i$ for all $(p_1,\ldots, p_n)\in\Omega^n$.

A pair of alternatives is called *trivial* if it always appears in the same order in all orderings, i.e., xpy for all $p\in\Omega$ or ypx for all $p\in\Omega$.

In order to construct a graph from a domain, we first have to form the triples that are permissible in the domain. Thus define

$$T(\Omega) = \{(xyz)\in A^3 : \exists p\in\Omega \text{ with } xpypz\}.$$

Whether a domain is nondictatorial (i.e., permits construction of a nondictatorial SWF) depends wholly on its triples as will be shortly shown. Moreover there might be different domains that have the same sets of triples and thus it is useful to consider equivalence classes of domains: define two domains Ω_1 and Ω_2 to be equivalent if $T(\Omega_1)=T(\Omega_2)$.

The following example shows that equivalence classes contain, in general, more than one member.

Let

$$\Omega_1 = \{(xyzw), (yxwz)\}$$

$$\Omega_2 = \{(xywz), (yxzw)\}.$$

Though $\Omega_1\cap\Omega_2=\phi$, they are equivalent since

$$T(\Omega_1) = T(\Omega_2) = \{(xyz), (yxz), (xzw), (xwz), (yzw), (ywz), (xyw), (yxw)\}.$$

Thus, two domains might be disjoint and equivalent.

The graph, denoted by $G(\Omega)$ is derived from $T(\Omega)$. To specify a graph we need to specify its nodes (or vertices) and branches (or edges). The *nodes* of $G(\Omega)$ are all the nontrivial pairs of A^2. The *branches* of $G(\Omega)$ are of three types[3]:

A *single branch* is an ordered pair of nodes [$(xy), (xz)$]. Thus it is a *directed* branch from the node (xy) to the node (xz).

A *joining branch* is an ordered set of nodes [$\{(xy), (yz)\}, \{(xz)\}$]. Thus it is a directed branch from the nodes (xy) and (yz) jointly to the node (xz).

A *splitting branch* is an ordered set of nodes [$\{(zx)\}, \{(yx), (zy)\}$]. Thus it is a directed branch which splits from the node (zx) to the nodes (yx) and (zy).

(*Continued*)

In our approach one has to check the given pool of preferences. If it is nondictatorial, then any vote, where the preferences are taken from this pool, results in a transitive outcome (according to some SWF).

Lastly, a refinement of this approach would be to consider domains of the type $\prod_i \Omega_i$ where $\Omega_i\subset\Sigma$, thus allowing for different "minority" types. Some work along this line is reported in Blair and Muller [1981].

[3] The existence of more than one type of branch causes the resulting creature to be known as a *hypergraph* (see Berge [1973]).

The relation between G and T is as follows:

if $(xyz), (yzx) \in T(\Omega)$ then $[(xy), (xz)], [(zx), (yx)] \in G(\Omega)$ and if $(xyz) \in T(\Omega)$ but $(yzx) \notin T(\Omega)$ then $[\{(xy), (yz)\}, \{xz\}], [\{(zx)\}, \{(yx), (zy)\}] \in G(\Omega)$. Thus a single branch is drawn from (xy) to (xz) and inversely from (zx) to (yx) if the triples (xyz) and (yzx) are permissible. If $(xyz) \in T(\Omega)$ but $(yzx) \notin T(\Omega)$ then a joining branch is drawn from (xy) and (yz) to (xz) and inversely a splitting branch is drawn from (zx) to (zy) and (yx). The graph thus is a concise summary of all the information needed to reconstruct the domain. The intuition becomes clear in the proof of Theorem 1: we draw a branch from one pair to another if whenever a coalition is decisive over the first pair is also decisive over the second.

A set of pairs is called *nontrivial* if it contains at least one nontrivial pair and is a proper subset of the set of all pairs in $A \times A$. Two sets S_1 and S_2, subsets of $A \times A$, are called *inverse complements* if both are nontrivial and for all nontrivial pairs.

$$(xy) \in S_1 \text{ if and only if } (yx) \notin S_2.$$

Define a set to be a *sink* if it is nontrivial and has no outgoing branches. That is, there is no branch connecting a pair inside a sink to a pair out of the sink. A joining branch extends out of the set S if $(xy), (yz) \in S$ but $(xz) \notin S$. A splitting branch extends out of S if $(zx) \in S$ but $(zy), (yx) \notin S$.

Define a domain to be nondictatorial (for any number of voters) if it permits construction of a SWF which is nondictatorial (regardless of the number of voters). Finally an equivalence class of domains is nondictatorial if all its members are nondictatorial.

The fact that the SWF is required to be nondictatorial for any number of voters is not restrictive since the existence of such a function is independent of the number of voters (see Section 4). Theorem 1 is a variant of the Kalai-Muller theorem [1977] and the proofs are close enough so that only a sketch of the proof is specified here.

THEOREM 1. *There exists a sink in the graph $G(\Omega)$ if and only if the equivalence class of Ω is nondictatorial.*

PROOF. Suppose a sink S exists in $G(\Omega)$. Define S^* to be the inverse complement of S, i.e., $S^* = \{(xy) \in A^2 : (yx) \notin S\}$. Define the following SWF: let one voter be decisive for the pairs in S, i.e., when he votes (xy) the outcome is (xy) regardless of the vote of the second voter. Let the second voter be decisive for the pairs of S^*. Let the coalition of the two voters be decisive on all pairs. If there are more voters we let them be dummies. It is cumbersome but rather straightforward to check that this function is indeed a well defined, transitive, nondictatorial Arrow SWF. The two sets S and S^* are sinks (S is a sink if and only if S^* is) which correspond to the sets R_1 and R_2 in Kalai-Muller's proof. Since the graph depends only on $T(\Omega)$, all the members of the equivalence class of Ω are non-dictatorial.

As for the reverse, suppose an equivalence class is nondictatorial. Thus any of its members admits a nondictatorial two person SWF. Define the set S to be all the pairs for which the first voter is decisive. We wish to show that S is a sink.

Suppose $(xy) \in S$ and $(xyz), (yzx) \in T(\Omega)$ we wish to show that $(xz) \in S$. In other words, let voter one be decisive for (xy). Is it possible to show that he is also decisive for (xz)? Let him vote (xyz) and let voter two vote (yzx). Those two orderings are permissible by assumption. The outcome is (xy) since voter one is decisive for (xy). The outcome is (yz) by the Pareto condition (both voters voted (yz)). Therefore, using *transitivity*, the outcome is (xz). Using IIA, the outcome of such a vote (one voting (xz) and two voting (zx)) is independent of the position of y. Thus, any time such a vote occurs, one prevails and the outcome is (xz). Therefore, one is decisive over (xz), since he voted (xz) while the other voter voted (zx). Thus $(xz) \in S$.

For a joining branch, in the same manner, one can check that if a coalition is decisive for (xy) and (yz) and $(xyz) \in T$, then it is decisive for (xz). For a splitting branch, if a coalition is decisive for (zx) and (xyz) is permissible then it is decisive *either* for (yx) *or* for (zy). Q.E.D.

Since we let voters be decisive on sinks only, the existence of a sink confines the individual to be decisive over certain pairs (and not all pairs). Thus, it defines his power and limits it. He is not a dummy (decisive only on trivial pairs) nor a dictator (decisive on all pairs).

Before turning to an example, two propositions are apparent now that the machinery is set up. First, we define two SWF's f_1 and f_2 to be *distinct* if there exist $P=(p_1,\ldots, p_n)$ and a pair $(xy) \in A^2$ such that $xf_1(P)y$ and $yf_2(P)x$. That is, two SWF's are distinct if under the same vote they yield a different result.

PROPOSITION 1. *The number of distinct two-person nondictatorial SWFs on a domain Ω is the number of distinct sinks in the graph of $T(\Omega)$.*

PROPOSITION 2. *Let Ω_1 and Ω_2 be two domains with $\Omega_1 \subset \Omega_2$. The number of distinct two-person nondictatorial SWFs in the smaller domain Ω_1 is at least as large as the number of distinct two person nondictatorial SWFs in the larger domain Ω_2, provided that the set of trivial pairs is the same in both domains.*

PROOF *of Proposition* 1. Given two distinct sinks S_1 and S_2 and their inverse-complements S_1^* and S_2^*, we can form two SWF's f_1 and f_2 by letting the two voters be decisive for the pairs in S_1 and S_2 respectively. Since $S_1 \neq S_2$, without loss of generality we can assume that there exists a pair $(xy) \in S_1$ and $(xy) \notin S_2$. Observe the profile P in which voter one votes (xy) and voter two (yx). f_1 and f_2 will differ on P since $xf_1(P)y$ but $yf_2(P)x$.

Thus for each sink S and its inverse complement S^* we can define two distinct SWF's if $S \neq S^*$ by switching the roles of voters one and two, and one SWF if $S=S^*$.

PROOF *of Proposition* 2. We have to show that every set which is a sink in the larger domain Ω_2 is also a sink in the smaller domain Ω_1. A set is a sink if it is nontrivial and has no outgoing branches. Since the set of trivials is the same in both domains then any nontrivial set in the larger domain is nontrivial in the smaller domain as well. As for branches, the number of single branches can only decrease with the decrease in the domain.

As for joining branches, *their number can increase but only where single branches existed before.* Consider a jointing branch (xy), (yz) to (xz), i.e., $(xyz) \in T(\Omega_1)$. If this is a newly formed branch then either (yzx) or $(zxy) \in T(\Omega_2)$. If (yzx) existed then there was a branch from (xy) to (xz). Thus the new branch cannot extend out of an existing sink since then this set could not have been a sink in the larger domain. If (zxy) existed, then there was a single branch from (yz) to (xz) and the same argument applies. The same is true with respect to splitting branches (note that when a single branch connects (xy) to (xz) then necessarily a single branch connects the inverses, i.e., connects (zx) to (yx)). It might be asked whether the proposition holds also without the provision for the number of trivial pairs. This is answered in the negative as can be seen from the next counterexample (taken from Kalai, Muller, Satterthwaite [1979]):

Let

$$\Omega_1 = \{(x_i x_j x_k y_l y_m y_n) \mid 1 \leq l, m, n, i, j, k \leq 3\}$$

$$\Omega_2 = \{(x_i x_j x_k y_1 y_2 y_3) \mid 1 \leq i, j, k \leq 3\}.$$

Since the x's and the y's are both free triples, there is a dictator on each and thus Ω_1 admits two, 2-person SWF's while Ω_2 admits none since all the pairs involving y's are trivial. Thus, $\Omega_1 \supset \Omega_2$ but the number of distinct SWF's in Ω_1 is larger than in Ω_2.

3. AN EXAMPLE

In this section we derive the graph for the domain given in the introduction.

A. *The Domain.*

$$\Omega = \{yzxw,\ zxyw,\ xwyz,\ ywzx,\ xyzw,\ zwxy,\ zxwy,\ xywz,\ yzwx\}.$$

From this domain, we wish to construct the graph, and so we have to specify the triples from which, by the method specified in the previous section, we can build the branches. The result is as follows:

B. *Triples* $(T(\Omega))$ *and Corresponding Branches.*

$$xyz,\ zxy,\ yzx \quad \begin{cases} (xy)\to(xz),\ (zx)\to(yx) \\ (zx)\to(zy),\ (yz)\to(xz) \\ (yz)\to(yx),\ (xy)\to(zy) \end{cases}$$

$$zxw,\ wzx,\ xzw \quad zwx,\ xwz \quad \begin{cases} (wz)\to(wx),\ (xw)\to(zw) \\ (xw)\to(xz),\ (zx)\to(wx) \\ (zx)\to(zw),\ (wz)\to(xz) \\ (xz)\to(xw),\ (wx)\to(zx) \end{cases}$$

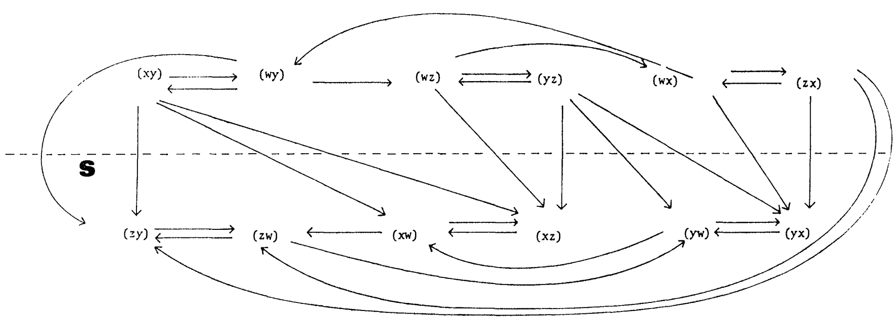
$$ywx,\ yxw,\ xyw \quad wxy,\ xwy \quad \begin{cases} (yw)\to(yx),\ (xy)\to(wy) \\ (xy)\to(xw),\ (wx)\to(yx) \\ (yx)\to(yw),\ (wy)\to(xy) \\ (wx)\to(wy),\ (yw)\to(xw) \end{cases}$$

$$ywz,\ yzw,\ zyw \quad wyz,\ zwy \quad \begin{cases} (zy)\to(zw),\ (wz)\to(yz) \\ (yz)\to(yw),\ (wy)\to(zy) \\ (wy)\to(wz),\ (zw)\to(yw) \\ (zw)\to(zy),\ (yz)\to(wz) \end{cases}$$

It is evident from observing the triples that the last three preferences in Ω are redundant, that is, the domain with the first six preferences only is equivalent to Ω.

C. *The Graph* $G(\Omega)$. From the specification of the branches, the graph can be constructed. From it ("Main Example's Graph") it is evident that $S=\{(zy), (zw), (xw), (xz), (yw), (yx)\}$ is a sink since no branch extends from this set to any pair outside the set. Indeed this is the only sink and thus the SWF which is constructed is unique.

Main Example's Graph



$S=\{(zy), (zw), (xw), (xz), (yw), (yx)\}$ is the only sink (with no branches extending out of the set). Therefore a unique SWF exists on this domain.

D. *The SWF.* The two-person *SWF* can be constructed as follows: We let one voter be decisive over all the pairs in S, and the other voter over the pairs in $S^* = \{(ab) \in A^2 : (ba) \notin S\}$. Since $S = S^*$ in this case, the SWF is such that the two voters are decisive over the same set S. This function satisfies the Pareto criterion, Independence of Irrelevant Alternatives, transitivity, and nondictatorship.

It should be noted that in this example there are only single branches. The other two types (which do not occur in this graph) play an important role in other cases. For example, suppose that the preference $(zywx)$ would be added to Ω. The only triple added to T would be zyx. No single branch would be added to the graph (since zyx could produce a single branch only with yxz or xzy, which are not in T). However, a joining branch would exist now from (zy) and (yx) to (zx) (and a splitting branch as well). Thus S would not be a sink since anyone who is decisive over any of its pairs would first be decisive over all its pairs, and second, because of the new joining branch, over all the pairs. Thus the new domain is dictatorial.

Returning to the original example, an n-person SWF can be constructed by making the rest of the n-2 voters dummies (i.e., their votes do not count). This function will have all the properties of the previous one including, obviously, nondictatorship. The question whether we can have a SWF with a more reasonable power distribution is addressed in the next section.

4. ANONYMOUS DOMAINS

A SWF is anonymous (symmetric in game theory terminology) if any permutation of the voters would not change the outcome. Thus, for example, there cannot be a tie-breaking chairman (with an extra vote when needed). Moreover two different coalitions of equal size cannot be decisive over different pairs of alternatives. A SWF is *monotonic* if whenever a coalition C is decisive over a pair, all its supercoalition $D \supseteq C$ are decisive over the pair as well.

The purpose of this section is to prove that the existence of an anonymous SWF is independent of the number of voters as long as it is more than three, and to characterize anonymous domains.

The question of adding or deleting voters is important for two reasons: First, it is easier to characterize domains that admit two or three persons SWF's. Second, since in most voting situations, some of the eligible voters do not vote then if we are assured of a well behaved voting mechanism for n voters, we would like to have a well behaved voting mechanism (even if different) for all $k \leq n$.

Kalai and Muller [1977] have proved the following result on the irrelevance of electorate size for the nondictatorial case:

THEOREM 1 (*Kalai-Muller*). *For $n \geq 2$, there exists a nondictatorial n-person SWF on Ω if and only if there exists a nondictatorial 2-person SWF on Ω.*

The extension of this theorem to the anonymous case is the following:

THEOREM 2[4]. *There exists an anonymous monotonic n-person SWF on Ω for all $n>3$ if and only if there exists an anonymous monotonic 3-person SWF on Ω.*

The proof of the theorem follows the proof of Lemma 3. The following counterexample shows that the theorem cannot be extended to two voters.

Consider the "main example" domain of Section 3. The SWF for two voters which was constructed for that domain makes the two voters individually decisive over the same set S. Thus this two-person SWF is anonymous. However, there does not exist a three-person anonymous SWF on this domain. To see this, denote by $S(i)$ the set of pairs for which voter i is decisive. Let $S(i, j)$ be the set of pairs for which i and j are *jointly decisive* and let P be the set of all pairs. Suppose that a three-person, anonymous SWF does exist on Ω, then clearly $S(i)=S(j)$ for all i, j. The set over which an individual is decisive is either empty or the set of all pairs or a sink. Thus there are three possibilities:

a. $S(i)=\phi$ for all i and so $S(j, k)=P$ for all $\{j, k\}\in\{1, 2, 3\}$. This function is majority rule which is not transitive since the domain includes (more than one) paradox of voting configuration.

b. $S(i)=P$ for all i. This function does not yield asymmetric preferences, (let 1 vote xy and 2 vote yx).

c. $S(i)=S$ for all i. This function is not transitive (let 1, 2 and 3 vote yx, xz and zy respectively).

Thus, this domain admits an anonymous two-person SWF but not a three-person anonymous SWF.

The main thrust of Theorem 3 is that under an anonymous SWF there must be one set of pairs of alternatives for which *all* coalitions are decisive. Under majority rule, on the other hand, the decisive coalitions are those that have more than a certain proportion of the voters — and they are decisive over all the alternatives. Majority rule, however, works for a limited variety of domains. If for some triple of alternatives, the configuration of preferences associated with the paradox of voting is permissible (i.e., (xyz), (yzx), $(zxy)\in T$) then majority rule need not be transitive over that triple. The question which is answered in this section is what are the necessary and sufficient conditions for some anonymous SWF to exist, not just majority rule. Indeed the occurrence of the paradox configuration affects the structure of the function but has no bearing on whether such a function exists.

The set over which all coalitions will be decisive is a certain type of a sink whose pairs can be ordered according to a transitive ordering.

A set S is *formed* out of the ordering $p\in\Sigma$ if $S=\{xy\in A\times A: xpy\}$. That is,

[4] Note that this theorem is not a straightforward extension of the nondictatorial case since we require that f be defined for all $n\geq 4$ and not just one given $n\geq 4$ for the 3-person function to exist. The converse is the same in both cases: if there exists a SWF for 3 (or 2 in the nondictatorial case) then there exists a SWF for all n.

its pairs are all the pairs that appear in the ordering.

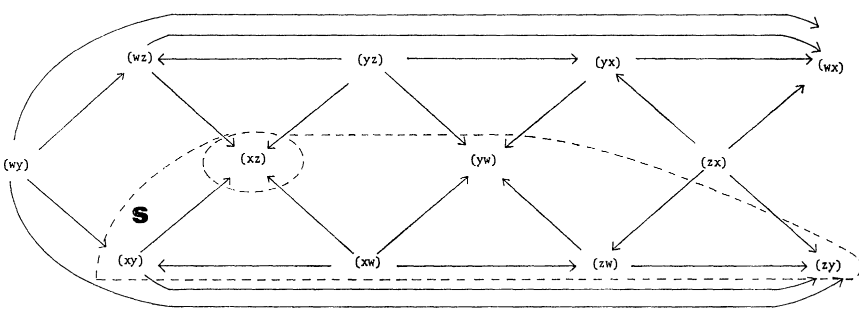
A *transitive sink* is a sink which is formed out of ordering $p \in \Sigma$. Thus, exactly half of the set of all pairs belongs to the transitive sink.

Define a domain to be *anonymous for any number of voters* if it permits construction of a family of monotonic anonmous SWF's, one for each different electorate size. Finally an equivalence class of domains is anonymous for any number of voters if all its members are anonymous for any number of voters.

THEOREM 3. *There exists a transitive sink in the graph $G(\Omega)$ if and only if the equivalence class of Ω is anonymous for any number of voters.*

Note that if we require an anonymous SWF for a *fixed* electorate the theorem does not imply the existence of a transitive sink. From the proof it becomes clear that the existence of a SWF for a *fixed* electorate whose size is a multiple of three implies the existence of a transitive sink. The main example domain of Section 3 shows that this is not the case when the electorate size is not a multiple of three. An anonymous two-person SWF exists on the domain Ω but $G(\Omega)$ does not contain a transitive sink since the unique sink S is not formed out of any ordering since it contains the pairs (zy) (yx) and (xz).

The Four Alternatives Paradox Graph



xz is a sink (no outgoing branches) and therefore a nondictatorial SWF exits.
S is a transitive sink (formed from the ordering $xzyw$) and therefore an anonymous SWF exits for any number of voters.

The theorem states that if a family of SWF's exists, one for each different electorate size, then and only then there exists a transitive sink.

LEMMA 1. *If a domain permits construction of a monotonic SWF, and if there exists a nonempty set of pairs over which three mutually disjoint and collectively exhaustive coalitions are decisive, then this set is a transitive sink (and in particular it contains exactly half of all the pairs.)*

PROOF. Denote by A_i, $i \in \{1, 2, 3\}$ the three coalitions which are decisive over the same set S. Since $A_i \cap A_j = \phi$ then clearly S cannot contain a pair and its inverse; otherwise the outcome would not be an ordering.

Since the SWF is well defined, if a coalition is decisive over a set S, its complement is decisive over the inverse complement $S^* = \{ab \in A \times A : ba \notin S\}$.

Suppose xy, $yz \in S$ but $xz \notin S$; then by definition $zx \in S^*$. Since each A_i is decisive over S, because of monotonicity, we have that $A_j = \overline{A_i \cup A_k}$ is decisive over S^* since $A_i \cup A_k$ is decisive over S. Let the members of A_1 vote xy, the members of A_2 vote yz, and the members of A_3 vote zx. The outcome is a cycle $xyzx$ — contradiction.

To show that S contains exactly half of the pairs, suppose xy, $yx \notin S$. Then xy, $yx \in S^*$ and we let A_1 vote xy, $A_2 - yx$ and so the outcome is xyx — contradiction.

LEMMA 2. *Consider the sets of pairs over which three mutually disjoint and collectively exhaustive coalitions are decisive. If these sets are empty and if the domain permits construction of a monotonic SWF, then the paradox configuration is not permissible.*

PROOF. Denote by A_i, $i \in \{1, 2, 3\}$ the three coalitions which are decisive over the empty set. From the definition of an inverse complement set, $\bar{A}_i$ is decisive over the set of all pairs, $i = 1, 2, 3$.

Suppose that the paradox configuration is permissible, i.e., xyz, yzx, $zxy \in T(\Omega)$ for some $x, y, z \in A$. Let $\bar{A}_1$ vote xy, $\bar{A}_2 yz$ and $\bar{A}_3 - zx$. Then $\bar{A}_1 \cap \bar{A}_2 = A_3$ voted xyz, $\bar{A}_2 \cap \bar{A}_3 = A_1$ voted yzx and $\bar{A}_1 \cap \bar{A}_3 = A_2$ voted zxy. The outcome is $xyzx$ — contradiction.

LEMMA 3. *If the paradox configuration is not permissible, then every set of pairs which is formed from an ordering belonging to Ω is a transitive sink.*

PROOF. We have to show that S which is formed from the ordering $p \in \Omega$ is a sink since transitivity is assured by definition.

To prove that the set is a sink consider a pair xy out of the ordering such that xpy and define the following sets of alternatives.

$$A_1 = \{a \in A \mid apx\}$$

$$A_2 = \{a \in A \mid xpa \text{ and } apy\}$$

$$A_3 = \{a \in A \mid ypa\}.$$

That is, the ordering p is such that elements of A_1 are preferred to x which is preferred to elements of A_2 which are preferred to y which, in turn, is preferred to A_3.

We want to show that a branch beginning at xy can end only at a pair of S. The three types of branches have to be considered:

A single branch can exist from xy to either xa or to ay. Consider the branch $xy \to xa$. Since $A_1 \cup A_2 \cup A_3 \cup \{x\} \cup \{y\} = A$, a has to belong to one of the A_i's.

If $a \in A_1$ then by the definition of branch type 1, xya, $yax \in T$, together with

$axy \in T$ (since $a \in A_1$) they form the paradox, a contradiction. If $a \in A_2 \cup A_3$ then $xa \in S$, and thus the branch from xy to xa ends at a pair of S.

Consider the branch $xy \to ay$. If $a \in A_1 \cup A_2$, then $ay \in S$. If $a \in A_3$ then yax, $axy \in T$, together with $xya \in T$ they form the paradox — contradiction.

A joining branch extends from S to a pair out of S if xy, $yz \in S$, $xyz \in T$ and $xz \notin S$. But if xy, $yz \in S$, then, since the pairs in S form an ordering p, $xz \in S$.

A splitting branch extends from S to a pair out of S if $xy \in S$, $yax \in T$ and both xa and $ay \notin S$. If $a \in A_1$ then $ay \in S$, if $a \in A_2 \cup A_3$, $xa \in s$.

PROOF *of Theorem* 2. (a) If the domain permits construction of an anonymous monotonic SWF for all $n \geq 4$, in particular it does so for n which is divisible by three.

Consider three disjoint coalitions of equal size $(n/3)$. They are mutually disjoint and exhaustive. Parts (c) and (d) of this proof show that there exists an anonymous SWF for any number of voters including three and two.

(b) If there exists an anonymous monotonic SWF for three voters, then consider the three voters as three coalitions. They are mutually disjoint, exhaustive and are of equal size. Parts (c) and (d) of this proof show that there exist an anonymous SWF for any number of voters.

(c) Because of anonymity, if one of the coalitions is decisive over a set, all three are decisive over the same set. Thus consider one of the coalitions, say A_1 and collect all the pairs on which it is decisive. If this set is not empty, then by Lemma 1 there exists a transitive sink. If the set is empty, then by Lemma 2 there does not exist a paradox configuration and by Lemma 3 there exists a transitive sink.

(d) If there exists a transitive sink S, then define the following SWF for any any number of voters:

Let all voters be individually decisive over all pairs in the set S, that is, if a voter votes xy and $xy \in S$ then the outcome is xy. We wish to show that this function is indeed a monotonic anonymous SWF, i.e., it is a well defined, monotonic, anonymous, transitive function satisfying the Pareto criterion and Independence of Irrelevant Alternatives. If a pair xy is *not* in S, then the outcome is xy iff all voters voted xy. Since S contains exactly half of the pairs, with no pair and its inverse, the SWF is well defined. Likewise, all the rest of the required properties follow directly from the definition except transitivity. Suppose, therefore, that a cycle $xyzx$ exists as an outcome of the SWF.

Case 1. xy, $yz \in S$. In this case $xz \in S$ and so all voters voted zx. In particular, someone voted zxy and someone yzx. The existence of these two implies the existence of the branch $yz \to yx$ and so $yx \in S$. This contradicts the fact that S is a transitive sink since $xy \in S$.

Case 2. $xy \in S$ *and* $yz \notin S$. Since $yz \notin S$, all voted yz. In particular someone voted xyz and someone yzx. Therefore a branch $xy \rightarrow xz$ exists. Thus $xz \in S$. Thus all voted zx, which contradicts the fact that someone voted xyz.

Case 3. $xy \notin S$ *and* $yz \in S$. The existence of the cycle $xyzx$ is equivalent to the existence of the cycle $yzxy$. Thus if $zx \in S$ Case 3 is equivalent, *mutatis mutandis*, to Case 1, and if $zx \notin S$ it is equivalent, *mutatis mutandis*, to Case 2.

Case 4. xy, $yz \notin S$. In this case all voters voted xy and yz and thus voted xyz. This contradicts the fact that the outcome is zx. Q.E.D.

PROOF *of Theorem* 3. (a) *necessity*: Suppose the domain permits construction of an anonymous monotonic SWF for all n. Using part a and c of the proof of Theorem 2 there exists a transitive sink.

(b) *sufficiency*: Suppose a transitive sink exists in the graph of $T(\Omega)$, part d of the proof of Theorem 2 shows that there exists an anonymous monotonic SWF on Ω. Q.E.D.

5. DISCUSSION

The social welfare function defined by a transitive sink is such that all voters are individually decisive for the pairs of the sink. This gives each one veto power over the half of the pairs in A^2 which are the inverses of the pairs of the sink. The more familiar method of majority rule works on a much more restricted domain. One can now ask the following — is it possible to combine the two methods — apply majority rule on triples for which the paradox configuration does not arise (i.e., at least one of xyz, yzx or zxy is not permissible) and a veto method for other triples? The answer is that such a rule can always be constructed whenever an anonymous rule exists.

Consider the following example, which is the four alternatives paradox.

$$\Omega = \{(xwyz), (wyzx), (yzxw), (zxwy)\}.$$

The paradox configuration occurs for each triple in this domain. Since the set S in the four alternatives paradox graph, which is formed from the ordering $xzyw$, is a transitive sink, we can let all individuals have veto power over all the inverses of the pairs of S and achieve a transitive outcome by Theorem 3.

However, a combined veto/majority rule can be constructed as follows: Let all individuals have veto power over zx and wy, i.e., each voter can block the outcome zx and wy by voting the inverses xz and yw, assuring the outcomes xz and yw. Apply majority rule for all other pairs except, of course, xz and yw over which voters are individually decisive. It is rather straightforward to check that this function will be transitive whenever the chosen sink S, over which voters have veto power, satisfies the following three conditions:

a. S does not contain a pair and its inverse
b. If xy, $yz \in S$ then $xz \in S$

c. Out of each triple which forms a paradox, at least one pair belongs to S. Since a transitive sink satisfies these conditions then whenever the domain permits construction of an anonymous SWF, the veto/majority rule will be transitive.

With respect to veto power a similar result was obtained in Blair and Pollak [1981]. They investigated the case where preferences are not restricted, but the requirement of transitivity is weakened to that of acyclicity. While in the transitive/full domain case an individual who can veto one pair of alternatives can veto any pair, both in their setting and in this restricted domain setting the power to veto one pair does not imply the power to veto every pair. Indeed what makes this last example work is that individuals can veto some but not necessarily all pairs of alternatives.

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