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Turnpike Properties of Capital Accumulation Games

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A differential game is considered in which players accumulate capital, their payoff functions depend upon the capital stocks of both players and their cost functions are convex. Previous existence and stability results are relied upon to show that the game, under an additional assumption, possesses the following properties: (a) Every equilibrium of the infinite horizon game converges to the unique stationary equilibrium. (b) For a time horizon long enough the finite horizon equilibrium stays in the neighborhood of the infinite horizon equilibrium except for some final time. (c) For a time horizon long enough the finite horizon equilibrium stays in the neighborhood of the stationary equilibrium except for some initial and final time. *Journal of Economic Literature* Classification Numbers: 022, 611. © 1986 Academic Press, Inc.

The asymptotic properties of optimal capital accumulation paths are usually referred to in the growth literature as "turnpike theorems." In our previous work [7] we studied a class of dynamic games in which capital is accumulated by the participating firms. The purpose of this paper is to investigate the turnpike properties of the equilibrium path of these capital accumulation games.

First, we specify the conditions under which every equilibrium path of the infinite horizon game converges to the unique stationary quilibrium. This property, which is usually referred to as global asymptotic stability, was investigated for capital acumulation growth models. See, for example,

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Cass and Shell [4] and Brock and Scheinkman [2]. This property implies that regardless of the initial stock of capital, the equilibrium path converges to a particular stationary point which does not depend on the initial conditions.

The second turnpike property describes the relation between the equilibrium pahts of the finite and infinite horizon games. Specifically, for a time horizon that is long enough, the finite horizon equilibrium path stays in an ε-neighborhood of the infinite horizon equilibrium, except for some final time.

In the third theorem we use the first and the second theorems to come up with the following turnpike property: For a time horizon that is long enough, the finite horizon equilibrium path stays in an ε neighborhood of the stationary equilibrium, except for some initial time required to accumulate capital and some final time in which "end game" considerations take over. This last result is an extension of the "balanced," or "modified golden rule" result of the optimal economic growth (see Cass [3]).

The three properties are depicted in Fig. 1. They are similar to McKenzie's [10] properties that were defined for a single decision maker's growth path. The main difference is in the second property. In McKenzie, the early turnpike path was close to the *stationary* path K^* for some initial period. In our analysis the early turnpike path is close to the infinite horizon path $K^\infty(t)$ for an initial period.

With respect to Theorem 1, there are two interesting results concerning asymptotic stability in differential games. The first by Brock [1] assumes

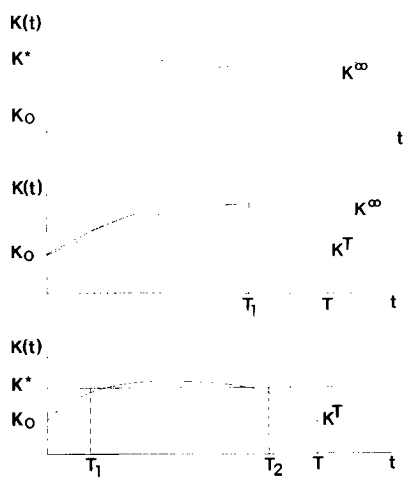

FIGURE 1

existence and shows the conditions for global asymptotic stability (GAS). His conditions, however, are more restrictive than ours (e.g., upper limit of the discount rate) for a more general model. In addition, what we show in our approach are the conditions for GAS (assumptions 5 and 6) over and above the conditions for existence (1)–(4) since we have separated the existence and GAS issues.

The second is by Haurie and Leitmann [9], who assume existence and uniqueness to a more general model, but with zero discount rate. Thus, the conditions are not compatible.

One interesting notion of stability that we have not dealt with, studied by Cheng and Hart [5], is the Cournot–Nash reaction function notion of stability, i.e., the stability of the Nash equilibrium path under small deviations.

The formulation follows our previous work [7]. We consider a game $G(K_0, T)$ where each player accumulates capital K_i according to

$$\dot{K}_i = I_i - \delta_i K_i,\ K_i(0) = K_{i0}, \qquad i = 1, 2 \tag{1}$$

where I_i is the investment in capital stock K_i of firm i, and δ_i is the depreciation constant.

The payoff for each player is its total discounted profits:

$$J_i = \int_0^T e^{-rt}\{\pi_i(K_1, K_2) - C_i(I_i)\}\, dt \tag{2}$$

where r is the discount rate, T might be finite or infinite, $\pi_i(K_1, K_2)$ is the instantaneous profit function, and $C_i(I_i)$ is the cost of investing I_i units.

Assumption 1. The control $I_i(t)$ takes its value in a compact set $[0, \bar{I}_i]$. A convex cost function $C_i(I_i)$ that satisfies that $C_i(I_i) \to \infty$ as $I_i \to \bar{I}_i$ will induce a control function satisfying assumption 1.

Assumption 2. $\pi_i(K_1, K_2) \in C^2$, is increasing and strictly concave function of K_i and decreasing in $K_j \cdot C_i(I_i) \in C^2$, is strictly increasing, strictly convex, and $C'_1(0) = 0$.

Assumption 3. $\pi_i^i = \partial \pi_i / \partial K_i$ is bounded, i.e., $|\pi_i^i| \leqslant L$ for some $L > 0$.

Assumption 4. $|\pi_i^{ij}| \leqslant L_i$ for some $L_i > 0$ and $C''_i > \varepsilon_i$ for some $\varepsilon_i > 0$.

Assumption 5. $\pi_1^{11}\pi_2^{22} > \pi_1^{12}\pi_2^{12}$ and $\pi_i^{12} \neq 0$ for $i = 1, 2$ and all K_1 and K_2.

We define a Nash equilibrium in path strategies and stationary Nash equilibrium as in our previous work, where we showed that under assumptions 1 through 4, for every initial conditions K_0, there exists a

Nash equilibrium for the game $G(K_0, T)$, for both finite and infinite T; under assumptions 1 through 5 there exists a unique stationary Nash equilibrium K^*, and *there exists* a Nash equilibrium that converges to K^*.

The assumption that yields the turnpike properties is the following:

Assumption 6. $|\pi_i^{ii}| \leqslant M_i$ for some $M_i > 0$ and $|\pi_i^{ii}| > |\pi_i^{ij}|$.

Note that this is a somewhat stronger assumption than assumptions 4 and 5. Specifically, assumption 6 implies the first parts of assumptions 4 and 5. In order to see the economic intuition of Assumption 6, assume that it does not hold so that $|\pi^{ii}| < |\pi^{ij}|$, the effects, therefore, of j's action on i's marginal profits are larger than the effects of i's own actions. Any action of j will result in a larger reaction of the rival which causes a "chain" reaction that diverges rather than converges. Indeed, in the proof of Theorem 1 we use exactly the "dampening" effects of the reverse condition $|\pi^{ii}| > |\pi^{ij}|$ to show that such "chain" reactions become smaller and smaller and converge to zero as time approaches infinity.

THEOREM 1. First Turnpike Property. *Let $K(t)$ be a Nash equilibrium of the* game $G(K_0, \infty)$. *Under assumptions* 1 *through* 6, $\lim_{t \to \infty} \|K(t) - K^*\| = 0$, *where K^* is the unique stationary equilibrium.*

Before proving the theorem, note that it implies that *every* solution of the capital accumulation game converges to the stationary equilibrium. This extends our previous results that showed *the existence* of such a converging solution.

Proof of Theorem 1. An equilibrium path $(I(t), K(t))$ has to satisfy the following necessary condition (see, e.g., Brock [1]): adjoin the constraint to the objective function to define the current value Hamiltonian H so that the necessary conditions are

$$\dot{\lambda}_i - r\lambda_i = -\partial H_i/\partial K_i = -\partial \pi_i/\partial K_i + \lambda_i \delta_i \tag{3}$$

$$\partial H_i/\partial I_i = 0 = -C_i'(I_i) + \lambda_i. \tag{4}$$

We divide the proof into two steps. In the first we assume that for both players there exists a time point from which the capital paths are monotonic. In the second step we assume that such a time point exists just for one player or does not exist at all.

Step 1. Assume there exists t^* such that $K_i(t)$, $i = 1, 2$, is monotonic for $t \in [t^*, \infty)$, i.e., either $\dot{K}_i(t) \geqslant 0$ for all $t \in [t^*, \infty)$ or $\dot{K}_i(t) \leqslant 0$ for all $t \in [t^*, \infty)$.

By standard arguments (e.g., Gould [8]), the equilibrium path $K(t)$ cannot tend to either zero or to the upper limit of K, i.e., $\bar{K}$. Therefore it converges to some level of $\tilde{K}$. It remains to be shown that $\tilde{K} = K^*$.

From the uniqueness of the stationary Nash equilibrium it follows that it is sufficient to show that $\dot{\tilde{I}} = \dot{\tilde{K}} = 0$. The solution of Eq. (3) for λ_i is given by

$$\lambda_i(t) = \left[\zeta_i - \int_0^t \pi_i^i(K_1(s), K_2(s))\, e^{-(r+\delta_i)s}\, ds \right] e^{(r+\delta_i)t}. \tag{5}$$

From the boundness of the marginal profit function π_i^i and since the equilibrium path $K(t)$ cannot tend to its upper limit of $\bar{K}$ it follows that $\lambda_i(t)$ is bounded therefore:

$$\zeta_i = \int_0^{\infty} \pi_i^i(K_1(s), K_2(s))\, e^{-(r+\delta_i)s}\, ds. \tag{6}$$

Note that the above argument guarantees that the transversality condition for the infinite horizon problem holds.

Using l'Hopital's rule we conclude that $\lim_{t \to \infty} \lambda_i(t) = \pi_i^i(\tilde{K}_1, \tilde{K}_2)/(r + \delta_i)$. Substituting this into Eq. (3), it is evident that $\dot{\tilde{\lambda}}_i = 0$. Moreover, Eq. (4) now guarantees that $\dot{\tilde{I}}_i = 0$. In the same fashion we can conclude that $\tilde{K}_i = \lim_{t \to \infty} K_i(t) = \tilde{I}_i/\delta_i$.

Step 2. Assume that for at least one player there does not exist t^* such that $\dot{K}_i \neq 0$ for $t \in [t^*, \infty)$. Differentiating Eq. (4) with respect to time, and substituting λ_i and $\dot{\lambda}_i$ from (3) and (4) yield the following equation:

$$C_i'' \dot{I}_i = (r + \delta_i)\, C_i' - \pi_i^i(K_1, K_2). \tag{7}$$

The analysis can now be represented by a phase diagram in the (I, K) space, where $\dot{K}_i = 0$ is given by $I_i = \delta_i K_i$ and $\dot{I}_i = 0$ is given by $(r + \delta_i)\, C_i'(I_i) = \pi_i^i(K_1, K_2)$. The $\dot{I}_i = 0$ curve is not stationary in the K_i, I_i space. Its movement depends on the signs of π_i^{ij} and $\dot{K}_j$. When the path $(K_i(t), I_i(t))$ is in the region in which $\dot{K}_i < 0$ (i.e., above the $\dot{K}_i = 0$ boundary) it cannot cross the $\dot{K}_i = 0$ boundary unless the $\dot{I}_i = 0$ boundary is below the path. This is evident in Fig. 2. In the same way, when the path is in the region in which $\dot{K}_i > 0$ it cannot cross the $\dot{K}_i = 0$ line unless the $\dot{I}_i = 0$ boundary is above the path. Before the path can cross the $\dot{K}_i = 0$ line again, the movement of the $\dot{I}_i = 0$ boundary has to change direction so as to be below the path before it intersects the $\dot{K}_i = 0$ line. Thus, if $K_i(t)$ has an infinite number of extremal points, $K_j(t)$ has an infinite number of extrema as well. Moreover, as the discussion above shows, the extremal points of $K_i(t)$ and $K_j(t)$ interlace, i.e., $\dot{K}_i(t)$ cannot change sign more than once without $\dot{K}_j(t)$ changing sign at least once.

For a given path $K_j(t)$, $j = 1, 2$, define a *cycle* $c(t_a, t^a)$ as the path of $K_j(t)$ between two consecutive extremal points that occur at t_a and t^a. Let the *amplitude* of a cycle be the difference between the maximum and the

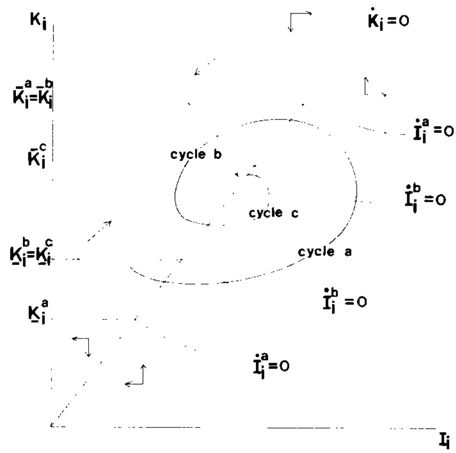

FIGURE 2

minimum of $K_j(t)$ in the cycle. From the previous discussion it is evident that the amplitude of a given cycle is bounded by the difference between the maximal and the minimal points of the intersection of $\dot{I}_i=0$ and $\dot{K}_i=0$. For example, the amplitude of cycle a in Fig. 2 is bounded by $\bar{K}_i^a-\underline{K}_i^a$ and similarly for cycles b and c whose amplitudes are bounded by $\bar{K}_i^b-\underline{K}_i^b$ and $\bar{K}_i^c-\underline{K}_i^c$, respectively. Let $c(t_\alpha, t^\alpha)$ and $c(t_\beta, t^\beta)$ be two consecutive cycles of $K_2(t)$, i.e., $t^\alpha=t_\beta$. From the previous interlacing argument, there exists a cycle $c(t_a, t^a)$ of $K_1(t)$ such that $t_a>t_\alpha$ and $t^a<t^\beta$. See, for example, Fig. 3.

We now claim that there exists an $\varepsilon>0$ such that

$$|K_1(t_a)-K_1(t^a)|<(1-\varepsilon)\max_{t_\alpha\leqslant t_1,t_2\leqslant t^\beta}|K_2(t_1)-K_2(t_2)|.$$

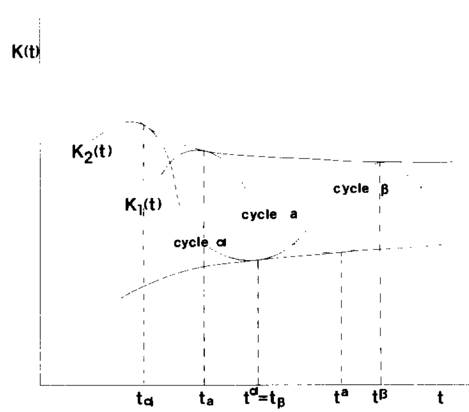

FIGURE 3

This will complete the proof since we have a *damped* series of cycles, i.e., an infinite number of cycles with a decrease in amplitude in each cycle. Moreover, since ε does not depend on the cycle, the amplitudes of the cycles approach zero as time tends to infinity and thus $K_i(t)$ converges for $i=1, 2$. By the argument of step 1, they converge to the unique stationary point. To show this last claim let $\hat{K}_1 = g(K_2)$ denote the level of capital at the intersection of the curve $\dot{I}_i = 0$ and the line $\dot{K}_i = 0$. From the previous discussion it is evident that $|K_1(t_a) - K_1(t^a)| < |g(K_2(t_a)) - g(K_2(t^a))|$. Observe that $g(K_2)$ is the solution of $\pi_1^1(K_1, K_2) =)(r+\delta_1)\, C_1'(\delta_1 K_1)$. Therefore,

$$|dg/dK_2| = |\pi_1^{12}|/|\pi_1^{11} - \delta_1(r+\delta_1)\, C_1''| < |\pi_1^{11}|/(\delta_1(r+\delta_1)\, C_1'' - \pi_1^{11})$$
$$= [1 + \delta_1(r+\delta_1)\, C_1''/(-\pi_1^{11})]^{-1} < 1 - \varepsilon$$

where $\varepsilon = (\delta_1(r+\delta_1)\, \varepsilon_1/M_1)/(1+\delta_1(r+\delta_1)\, \varepsilon_1/M_1)$, and ε_1 and M_1 are given in assumptions 4 and 6. Since $[t_a, t^a] \subset [t_\alpha, t^\beta]$ it follows that

$$|g(K_2(t_a)) - g(K_2(t^a))| \leqslant \underset{t_\alpha \leqslant t_1, t_2 \leqslant t^\beta}{\operatorname{Max}} |g(K_2(t_1)) - g(K_2(t_2))|.$$

Since g is a continuous function on a compact set $[t_\alpha, t^\beta]$ it achieves a maximum and minimum at times $\bar{t}$ and $\underline{t}$ respectively.

By the mean value theorem, there exists a mean value ϕ such that

$$|g(K_2(\bar{t})) - g(K_2(\underline{t}))| = |g'(\phi)|\; |(K_2(\bar{t}) - K_2(\underline{t}))|$$
$$< (1-\varepsilon)\, |K_2(\bar{t}) - K_2(\underline{t})|$$
$$\leqslant (1-\varepsilon) \underset{t_\alpha \leqslant t_1, t_2 \leqslant t^\beta}{\operatorname{Max}} |K_2(t_1) - K_2(t_2)|. \quad \text{Q.E.D.}$$

It should be noted that a similar though longer proof can be used to prove Theorem 1, with the following assumption replacing assumption 6:

Assumption 6'. $|\pi_i^{ii}| \leqslant M_i$ for some $M_i > 0$ and $\Pi_1^{12}\Pi_2^{12} > 0$.

Let $B_{Li}([0, T])$ be a family of continuous functions on $[0, T]$ that are bounded by a common bound and have the same Liptschitz constant. For all x_n, x_0, $\in B_{Li}([0, \infty))$, let $x_n \to^* x_0$ iff for every finite T $\operatorname{Sup}_{t \leqslant T} |x_n(t) - x_0(t)| \to 0$ as $n \to \infty$.

THEOREM 2. (Second Turnpike Property). *Let $K_T(t)$ be a Nash equilibrium for the game $G(K_0, T)$. Under assumptions* 1 *through* 6, *for a given $\varepsilon > 0$, for every $T_1 > 0$ there is a $T_2(T_1)$ such that for every $T > T_2(T_1)$ every solution $K_T(t)$ of the game $G(K_0, T)$ satisfies*

$$\underset{0 \leqslant t \leqslant T_1}{\operatorname{Sup}} \|K^\infty(t) - K_T(t)\| \leqslant \varepsilon.$$

For some solution $K^{\infty}(t)$ of the infinite horizon game $G(K_0, \infty)$.

Proof. Step 1. Assume, *a contrario*, that there exists T_1 and ε for which no T_2 exists as required. Therefore there is an infinite sequence $T_n \to \infty$ and K_{T_n} such that

$$\operatorname*{Sup}_{0 \leqslant t \leqslant T_1} \|K^{\infty}(t) - K_{T_n}(t)\| > \varepsilon \tag{8}$$

for every solution K^{∞} of $G(K_0, \infty)$. Since $B_{L^i}([0, T])$ is a compact set for every T, without loss of generality (taking subsequences if necessary), we can assume that $K_{T_n} \to^* J$. In step 2 we show that J is a solution of the infinite horizon game, which contradicts (8).

Step 2. Substituting Eq. (5) into (1) and solving for K^{∞} yields that K^{∞} satisfies the following equation:

$$K_i^{\infty}(t) = \xi + \int_0^t e^{-\delta_i(t-s)}(C_i')^{-1} \left\{ \int_s^{\infty} \pi_i^i(K_1^{\infty}(\tau), K_2^{\infty}(\tau))\, e^{-(r+\delta_i)(\tau - s)}\, d\tau \right\} ds \tag{9}$$

where $\xi = K_{i0} e^{-\delta_i t}$, and similarly for K_{iT_n}. Because of our guaranteed sufficiency (strict convexity of C and concavity of π) any pair of functions that satisfies (9) for $i = 1, 2$ is a Nash equilibrium for the game $G(K_0, \infty)$. Observe the following expressions:

$$\text{(a)} \qquad \int_0^t e^{-\delta_i(t-s)}(C_i')^{-1} \left\{ \int_s^{T_n} \pi_i^i(J_1(\tau), J_2(\tau))\, e^{-(r+\delta_i)(\tau-s)}\, d\tau \right\} ds + \xi$$

$$\text{(b)} \qquad \int_0^t e^{-\delta_i(t-s)}(C_i')^{-1} \left\{ \int_s^{\infty} \pi_i^i(J_1(\tau), J_2(\tau))\, e^{-(r+\delta_i)(\tau-s)}\, d\tau \right\} ds + \xi$$

where $J(\tau)$ is the value of the function J (the limit of K_{T_n}) at time τ. For a given t, the difference between (a) and (b) tends to zero as $n \to \infty$.

Next observe the following expressions:

$$\text{(c)} \qquad J_i - \left(\int_0^t e^{-\delta_i(t-s)}(C_i')^{-1} \left\{ \int_s^{T_n} \pi_i^i(J_1, J_2)\, e^{-(r+\delta_i)(\tau-s)}\, d\tau \right\} ds + \xi \right)$$

$$\text{(d)} \qquad K_{iT_n} - \left(\int_0^t e^{-\delta_i(t-s)}(C_i')^{-1} \left\{ \int_s^{T_n} \pi_i^i(K_{1T_n}, K_{2T_n})\, e^{-(r+\delta_i)(\tau-s)}\, d\tau \right\} ds + \xi \right).$$

The difference between (c) and (d) tends to zero as $n \to \infty$. This is true since $K_{iT_n} \to^* J_i$ and by assumptions 3 to 6, π_i^i, $[(C_1')^{-1}]'$ and π_i^{ii} and π_i^{ij}

are bounded. Since (d) is identically zero, for any given t, by definition of K_{iT_n} it follows that expression (c) tends to zero when $n \to \infty$. Now observe that the second term in (c) tends to J_i and to (b) as $n \to \infty$. Therefore J_i satisfies

$$J_i = \int_0^t e^{-\delta_i(t-s)}(C_i')^{-1}\left\{\int_s^\infty \pi_i^i(J_1, J_2)e^{-(r+\delta_i)(\tau-s)}\,d\tau\right\} ds + \xi. \tag{10}$$

It follows therefore that J is a Nash equilibrium for the game $G(K_0, \infty)$.

Q.E.D.

DEFINITION. Let $\Delta_T \subset B_{L_1}([0, T]) \times B_{L_2}([0, T])$ be the set of all capital paths that constitute a Nash equilibrium for the finite horizon game $G(K_0, T)$. Similarly, let $\Delta \subset B_{L_1}([0, \infty)) \times B_{L_2}([0, \infty))$ be the set of all capital paths that constitute a Nash equilibrium for the infinite horizon game $G(K_0, \infty)$.

ASSUMPTION 7. *The set Δ of Nash equilibria of $G(K_0, \infty)$ is finite.*

THEOREM 3. (Third Turnpike Property). *Let $K_T(t)$ be a Nash equilibrium for the game $G(K_0, T)$. Under assumptions 1 through 7, for every $\varepsilon > 0$ there exists T_1 such that* for every $T_2 > T_1$ *there is $\hat{T}$ for which for all $T > \hat{T}$:*

$$\operatorname*{Sup}_{K_T \in \Delta_T} \operatorname*{Sup}_{T_1 \leqslant t \leqslant T_2} \|K^* - K_T(t)\| < \varepsilon. \tag{11}$$

Note that although the stationary equilibrium K^* is unique, the equilibrium paths for the finite and infinite horizon games are not necessarily unique. The modifications in the extension of the turnpike theorem by Cass are made precisely for this reason. The thrust behind the proof is a combination of Theorem 1 and 2. Since Theorem 1 guarantees that every infinite horizon solution path converges to K^* and Theorem 2 implies that the finite horizon solution (for a long time horizon) is close to the infinite one, it follows that the finite horizon solution path has to be in a neighborhood of K^* for a sufficiently long time horizon.

Proof. Theorem 1 implies that for every $K_\infty \in \Delta$ there exist $T(K_\infty)$ such that

$$\operatorname*{Sup}_{t \geqslant T(K_\infty)} \|K_\infty(t) - K^*\| < \varepsilon/2. \tag{12}$$

Let $T_1 = \operatorname{Max}_{K_\infty \in \Delta} T(K_\infty)$. The assumption that Δ is finite guarantees that T_1 is finite. T_1 satisfies the following inequality:

$$\operatorname*{Sup}_{K_\infty \in \Delta} \operatorname*{Sup}_{t > T_1} \|K_\infty(t) - K^*\| < \varepsilon/2. \tag{13}$$

Theorem 2 guarantees that for every $T_2 > T_1$ we can choose $\hat{T}$ such that for every $T > \hat{T}$ and $K_T \in \Delta_T$ there is $K_\infty \in \Delta$ such that

$$\operatorname*{Sup}_{0 \leqslant t \leqslant T_2} \|K_\infty(t) - K_T(t)\| < \varepsilon/2. \tag{14}$$

For such K_T it follows from the triangular inequality that for every $K_\infty(t)$ (15) holds

$$\operatorname*{Sup}_{T_1 \leqslant t \leqslant T_2} \|K^* - K_T(t)\| \leqslant \operatorname*{Sup}_{T_1 \leqslant t \leqslant T_2} \|K^* - K_\infty(t)\| + \operatorname*{Sup}_{T_1 \leqslant t \leqslant T_2} \|K_\infty(t) - K_T(t)\|. \tag{15}$$

In particular, choose $K_\infty(t)$ such that (14) holds for $t \leqslant T_2$. Since (13) holds for every $K_\infty(t)$ for $t > T_1$ therefore the following holds:

$$\operatorname*{Sup}_{T_1 \leqslant t \leqslant T_2} \|K^* - K_T(t)\| < \varepsilon. \tag{16}$$

This completes the proof since (16) holds for every $K_T \in \Delta_T$ as long as $T > \hat{T}$. Q.E.D.

Three remarks are worth mentioning at this point:

(a) The assumption of a finite number of equilibria is essential for the existence of $T_1 = \operatorname{Max}_{K_\infty \in \Delta} T(K_\infty)$. Another assumption that can replace it s that K_∞ converge *uniformly* to K^*.

(b) Note that for T_2 *as large as we want* we can find $\hat{T}$ such that for time horizons larger than $\hat{T}$, the equilibrium path K_T is in the ε-neighborhood of the stationary equilibrium K^* between T_1 and T_2. Thus, by choosing a large enough time horizon, we have complete control over the time during which the finite horizon solution stays near the stationary equilibrium.

(c) The turnpike property is satisfied *uniformly* on Δ_T. Thus for appropriate T_1 and T_2 *all* the equilibrum paths $K_T \in \Delta_T$ for T large enough are in the ε-neighborhood of K^* for t between T_1 and T_2.

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