TECHNOLOGICAL FORECASTING AND SOCIAL CHANGE 20, 199–213 (1981)

A Nonsymmetric Responding Logistic Model for Forecasting Technological Substitution

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ABSTRACT

This article presents a simple model of technological substitution termed as nonsymmetric responding logistic (NSRL). Based on the theory that substitution is an imitation process, the model can accommodate different patterns of technological substitution by allowing the imitation effect to vary over time systematically. It allows the S-curve to be symmetrical as well as nonsymmetrical, with the point of inflection responding to the substitution process. Data from four medical innovations are analyzed to illustrate the generality of the model.

Introduction

Over the years, a number of mathematical models have been proposed to represent the time pattern of the substitution process, that is, the process by which the adoption of a new product or technology spreads and grows to replace an existing product or technology [8,10,12,13,15]. Modeling efforts have generally resulted in deterministic interpretation of the time-dependent aspects of the substitution process—the S-shaped or Pearl curve [10]. The underlying behavioral theory in the development of these models is that the substitution is an imitation process [14]. The process, in general, is described by following differential equation:

$$\frac{df}{dt} = bf\,(F - f) \tag{1}$$

where f is the market share of a product at time t, F is the upper limit of the market share, and the parameter b is the coefficient of imitation or internal influence relating the word-of-mouth communication or imitation between the adopters and potential adopters to the rate of adoption. The solution of equation (1) yields the S-shaped depiction of the substitution process.

Equation (1) has served as the basis of analysis for the models suggested by Mansfield [14], Blackman [2], and Fisher and Pry [6]. More specifically, Blackman and

This research was supported by the Center for Marketing Strategy Research, The Wharton School, University of Pennsylvania.

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0040-1625/81/08019915$2.75

Fisher-Pry models are solutions of equation (1), with the Fisher-Pry model considering the solution with $F = 1$. In addition, Floyd [7] has suggested addition of a term to the Blackman/Fisher-Pry model to improve the fit and forecasting of technological substitution. Sharif and Kabir [20] have proposed that a smooth, S-shaped curve within the region bounded by the Blackman/Fisher-Pry and Floyd curves can be obtained by combining these two models. More specifically, the following equations represent these models:

Blackman

$$\ln \frac{f}{F-f} = c_0 + bFt$$
$$= c_0 + ct \tag{2}$$

Fisher-Pry

$$\ln \frac{f}{1-f} = c_1 + bt \tag{3}$$

Floyd

$$\ln \frac{f}{F-f} + \frac{F}{F-f} = c_2 + ct \tag{4}$$

Sharif-Kabir

$$\ln \frac{f}{F-f} + \frac{\sigma F}{F-f} = c_3 + ct \tag{5}$$

where $c = bF$, c_0, c_1, c_2, and c_3 are constants and $0 \leqslant \sigma \leqslant 1$. The Sharif-Kabir model yields the Floyd model when $\sigma = 1$ and the Blackman/Fisher-Pry models when $\sigma = 0$. As derived in Appendix A, the following differential equation describes the Sharif-Kabir model:

$$\frac{df}{dt} = \frac{c}{F} \frac{f(F-f)^2}{F - f(1-\sigma)} \tag{6}$$

$$= \left[\frac{c(F-f)}{F(F-f(1-\sigma))}\right] \cdot f \cdot (F-f). \tag{7}$$

Substitution of $\sigma = 1$ into equation (7) provides the differential equation representing the Floyd model. That is,

$$\frac{df}{dt} = \left[\frac{c(F-f)}{F^2}\right] \cdot f \cdot (F-f). \tag{8}$$

Note that as compared to the Blackman/Fisher-Pry models, Floyd/Sharif-Kabir models essentially consider the coefficient of imitation as varying over time systematically. In fact, the Floyd model assumes implicitly that

$$b(t) = \frac{c(F-f)}{F^2}$$
$$= b\left(1 - \frac{f}{F}\right). \tag{9}$$

Similarly, the Sharif-Kabir model assumes implicitly the following specification for the coefficient of imitation:

$$\begin{aligned} b(t) &= \frac{c(F - f)}{F(F - f(1 - \sigma))} \\ &= b \frac{1 - (f/F)}{1 - \frac{f}{F}(1 - \sigma)}. \end{aligned} \tag{10}$$

However, in both the Floyd and Sharif-Kabir models, the coefficient of imitation can only decrease with time. For the Floyd model, this is clear from inspection of equation (9). For the Sharif-Kabir model differentiating of equation (10) with respect to f gives:

$$\frac{db(t)}{df} = \frac{-b\sigma}{F\left(1 - \frac{f}{F}(1 - \sigma)\right)^2}.$$

Because b is positive and $0 < \sigma < 1$, $b(t)$ must decrease with time. In fact, for both the Floyd and Sharif-Kabir models, $b(t)$ becomes zero at the full penetration, i.e., when $f = F$.

The implicit time-dependent specification of the coefficient of imitation in Floyd/Sharif-Kabir models is consistent with the argument made by Bretschneider and Mahajan [3] that because of the changing characteristics of the potential adopter population, technological changes, product modifications, pricing changes, general economic conditions, and other exogenous and endogenous factors, the diffusion model parameters are more likely to change over time. In fact, Kotler has also suggested that for some innovations the coefficient of imitation "should decline through time, rather than stay constant, because the remaining potential adopters are less responsive to the product and associated communications" [9, p. 531].[1]

However, Bundgaard-Nielsen [4] has argued that later adopters may adopt faster than earlier ones, provided information concerning the innovation is available to all, because "late adopters are in a better position to assess the new technology than earlier ones" [4, p. 366]. Consequently, the coefficient of imitation may be required to increase or decrease with time and the nonincreasing constraint of the Floyd/Sharif-Kabir models may be unduly restrictive.

The additional differences between the models, as also pointed out by Sharif and Islam [19], are in terms of two mathematical properties of the substitution curves generated by the models—point of inflection and symmetry. The point of inflection represents that stage of the substitution process at which the maximum rate of adoption, df/dt, is reached. It can be obtained by differentiating the rate of adoption equation (such as equations (1), (7), and (8)) with respect to f and solving the resulting equation, by setting it equal to zero, for f. As derived in Appendix B, the point of inflection for the Sharif-Kabir model is given by the solution to the following equation:

$$f^2(2 - 2\sigma) - 3fF + F^2 = 0. \tag{11}$$

When $\sigma = 1$, equation (11) yields $f = F/3$. Hence, by the Floyd model, the maximum

[1] For a brief overview of time-varying parameter estimation approaches, see Mahajan et al. [11].

rate of substitution is reached when the total adoption is about 33.3% of the upper limit of the market share. In general, the solution of equation (11) is given by

$$f = \frac{3F - \sqrt{F^2 + 8\sigma F^2}}{4(1 - \sigma)}. \tag{12}$$

When $\sigma = 0$, equation (12) yields $f = F/2$, which is the point of inflection for the Blackman/Fisher-Pry models. In other words, whereas the point of inflection is fixed for the Blackman/Fisher-Pry and Floyd models, the Sharif-Kabir model does respond to the substitution process. However, the range of the point of inflection offered by the Sharif-Kabir model is limited between $F/2$ and $F/3$. The implications of these results are rather restrictive. The Blackman/Fisher-Pry models, a priori, specify that

$$\left(\frac{df}{dt}\right)_{\max} = b\,\frac{F^2}{4} \tag{13}$$

and the Floyd model, a priori, assumes that

$$\left(\frac{df}{dt}\right)_{\max} = b\left(\frac{4F^2}{27}\right) \tag{14}$$

whereas the Sharif-Kabir model assumes a value between $bF^2/4$ and $4bF^2/27$. In other words, these models, a priori, assume the value, or a limited range of the values, for the maximum market share that can be achieved by a new product or technology in any single period, determined primarily by the coefficient of imitation.

In order to examine the symmetry property of these models, consider the value of the rate of adoption, df/dt, generated by these models around the inflection point, i.e., $f = f^* \pm \epsilon$ where f^* is the point of inflection. The Blackman/Fisher-Pry models yield

$$\frac{df}{dt} = b(F^2/4 - \epsilon^2) \tag{15}$$

for $f = ½ + \epsilon$ as well as for $f = ½ - \epsilon$. In other words, the rate of adoption is symmetric around the point of inflection. The Floyd model as well as the Sharif-Kabir models, however, are nonsymmetric. For example, for the Floyd model

$$\frac{df}{dt} = b\left(\frac{4}{27}F^2 + \frac{\epsilon^3}{F} - \epsilon^2\right) \tag{16}$$

for $f = F/3 + \epsilon$ and

$$\frac{df}{dt} = b\left(\frac{4}{27}F^2 - \frac{\epsilon^3}{F} - \epsilon^2\right) \tag{17}$$

for $f = F/3 - \epsilon$, indicating the nonsymmetric nature of the curve.

To conclude, the Blackman/Fisher-Pry models assume that the coefficient of imitation is time-invariant, the substitution curve is symmetric, and the point of inflection is fixed. The Floyd model assumes implicitly a time-decreasing coefficient of imitation and a nonsymmetric curve with a fixed point of inflection. The Sharif-Kabir model specifies implicitly a time-decreasing coefficient of imitation with a substitution curve that is nonsymmetric and responsive to the substitution process in a limited fashion.

The issue concerning the desirability of having a model that can accommodate different patterns of technological substitution by allowing the S-curve to be symmetrical as well as nonsymmetrical, with the point of inflection responding to the substitution process, has been raised by Sharif and Islam [19]. Commenting on the weaknesses of the existing models, they write:

> It is therefore not difficult to understand why each of these models shows remarkable success in describing certain situations and yet fails in other cases. In real-world situations the S-curve can by symmetric as well as nonsymmetric, with the point of inflection in either the earlier or later phase of development.

In fact, Sharif and Islam [19] suggest the Weibull distribution as a general model for forecasting technological change. This curve provides nonsymmetry with the point of inflection that responds to the substitution process. The idea to describe the new product growth via the Weibull distribution has been suggested and illustrated earlier by Pessemier [16]. However, the model, although possessing the desirable mathematical properties, ignores the underlying theory behind the substitution process. Furthermore, the behavioral and managerial interpretations of the parameters characterizing the model are not clear.

This article presents a simple model that overcomes these limitations. The underlying behavioral theory is that substitution is an imitation process [14]. The model, termed as *nonsymmetric responding logistic* (NSRL), can accommodate different patterns of substitution by allowing the coefficient of imitation to vary over time systematically. It allows the S-curve to be symmetrical as well as nonsymmetrical, with the point of inflection responding to the substitution process.

First, the derivation and the associated properties of the nonsymmetric responding logistic model are presented. Next, to illustrate the generality of the model, data from four medical innovations are analyzed.

Nonsymmetric Responding Logistic Model

Like the Floyd [7] and the Sharif and Kabir [20] models, the underlying rationale behind the proposed model is that the coefficient of imitation in equation (1) changes over time systematically and that this systematic variation can be expressed as a function of the penetration or the market share gained by the product at time t. That is,

$$b(t) = bf^{\alpha}, \tag{18}$$

where α is a constant. Substitution of equation (18) into equation (1) yields

$$\begin{aligned}\frac{df}{dt} &= bf^{\alpha+1}\,(F - f)\\ &= bf^{\delta}\,(F - f)\end{aligned} \tag{19}$$

where $\delta = \alpha + 1$. As compared to equation (1) or the Blackman/Fisher-Pry models, equation (19) includes a nonlinear interaction term to represent the word-of-mouth communication between adopters and nonadopters. If it is assumed that the substitution takes place without any influence, $\delta = 0$. However, presence of imitation in the substitution process is indicated by $\delta > 0$. The term is referred to as the *nonuniform influence factor.* Note that

$$\frac{db(t)}{df} = b(\delta - 1)f^{\delta-2}.$$

Thus

$$\frac{db}{df} < 0 \quad \text{for} \quad 0 < \delta < 1$$

$$\frac{db}{df} = 0 \quad \text{for} \quad \delta = 1$$

$$\frac{db}{df} > 0 \quad \text{for} \quad \delta > 1.$$

So the coefficient of imitation can increase or decrease with time or remain constant, a flexibility not offered by the other models. Its value at full penetration is b.

Note also that, compared to the Floyd/Sharif-Kabir models in equations (7) through (10), the specification suggested in equation (18) to represent the time-varying nature of the coefficient of imitation is simpler and easily interpretable. Furthermore, it can be easily shown that the substitution curve implied by equation (19) is nonsymmetric, with the point of inflection responding to the substitution process. In fact, the point of inflection is:

$$\frac{d}{df}\left(bf^{\delta}\left(F - f\right)\right) = 0$$

or

$$b\delta f^{\delta-1}\left(F - f\right) - bf^{\delta} = 0$$

or

$$\delta(F - f) - f = 0$$

or

$$f = \frac{\delta F}{1 + \delta} = \frac{F}{1 + \frac{1}{\delta}}. \tag{20}$$

As is clear from equation (20), unlike the Sharif-Kabir model, the range for the point of inflection is not fixed. In fact, as $\delta \to \infty, f \to F$ and as $\delta \to 0, f \to 0$. In other words, the nonuniform influence factor δ allows complete flexibility for the curve to respond to the substitution process. The effect of δ on the substitution curves is depicted in Figure 1.

The nonsymmetrical aspect of the model can be shown by substituting a value of $f = f^* \pm \epsilon$, where f^* is the point of inflection, in equation (19).

For $f = f^* + \epsilon$,

$$\frac{df}{dt} = b\left(\frac{\delta F}{1 + \delta} + \epsilon\right)^{\delta}\left(F - \frac{\delta F}{1 + \delta} - \epsilon\right)$$

$$= b\left(\frac{\delta F}{1 + \delta} + \epsilon\right)^{\delta}\left(\frac{F}{1 + \delta} - \epsilon\right). \tag{21}$$

Similarly, for $f = f^* - \epsilon$,

$$\frac{df}{dt} = b\left(\frac{\delta F}{1 + \delta} - \epsilon\right)^{\delta}\left(\frac{F}{1 + \delta} + \epsilon\right). \tag{22}$$

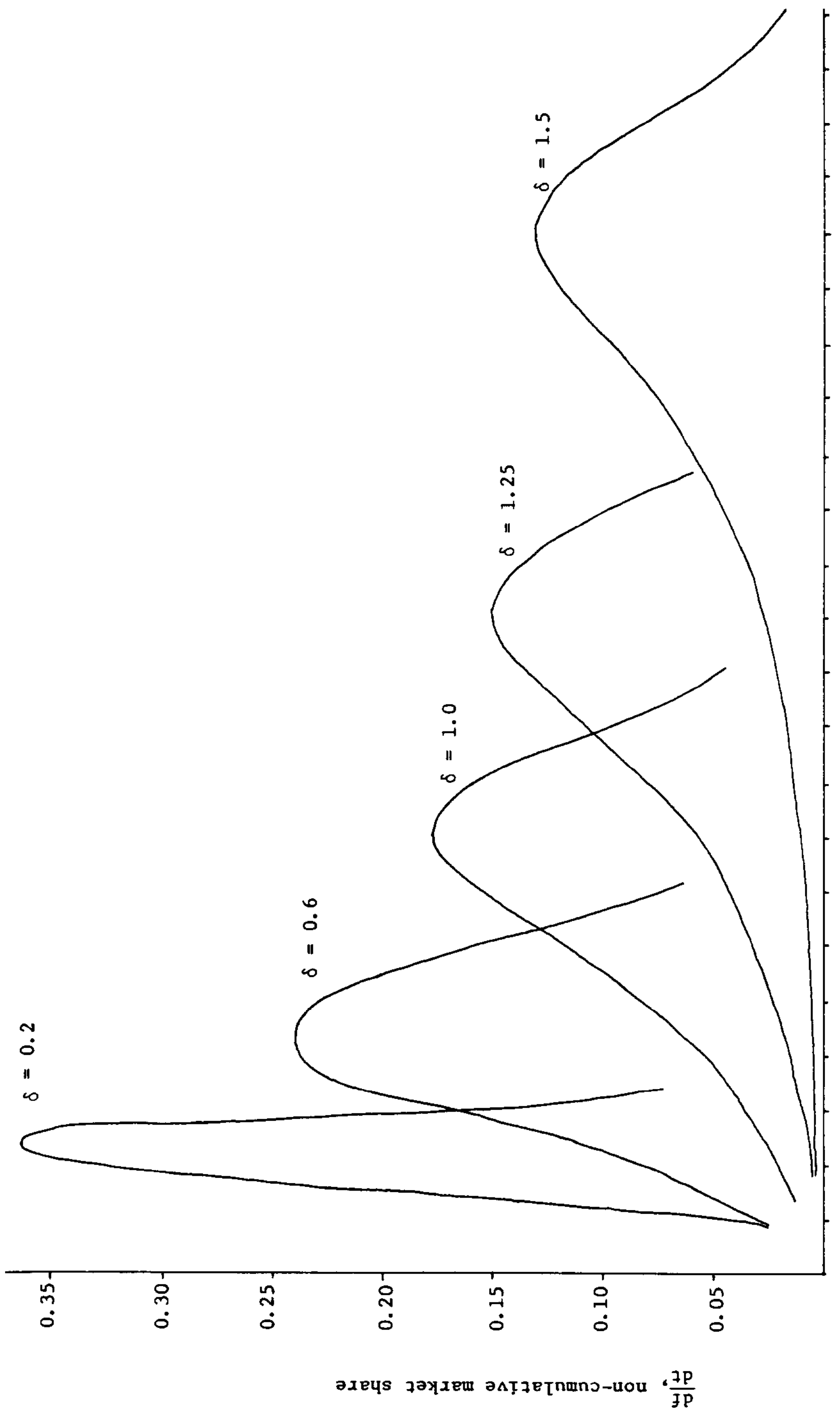

Fig. 1. Time effect of the nonuniform influence factor, δ, on substitution curves. Initial period market share is 3% and b = 0.7.

By taking different values of δ, it can be easily shown that equations (21) and (22) are not equal for $\delta = 1$. Hence, the curve is nonsymmetric.

Unlike the Blackman/Fisher-Pry and the Floyd models, the maximum market share that can be achieved in any period, using equations (13) and (14), is also not fixed and in fact depends upon the value of δ. That is,

$$\left(\frac{df}{dt}\right)_{max} = b\left(\frac{\delta F}{1 + \delta}\right)^{\delta}\left(\frac{F}{1 + \delta}\right). \tag{23}$$

Furthermore, when $F = 1$, it can be shown that (see [5]) the maximum market share that can be achieved by a new product or technology in any single period can vary between 0 and b. On the other hand, this value is $0.25b$ for the Blackman/Fisher-Pry model, $0.1481b$ for the Floyd model, equations (13) and (14), respectively, and between $0.1481b$ to $0.25b$ for the Sharif-Kabir model.

In short, equation (19) suggests a simple model of technological substitution that considers the coefficient of imitation as changing over time systematically. It is nonsymmetric, with the point of inflection responding to the substitution process. Hence, this model is termed as NSRL—nonsymmetric responding logistic. Unfortunately, a general closed form solution of equation (19) is not obtainable. However, this equation is solvable for certain values of δ [5]. The values of b, δ, and F in equation (19) can be estimated through any appropriate nonlinear regression analysis algorithm, by using the discrete analog of equation (19). That is,

$$f(t + 1) - f(t) = bf^{\delta}(t)\,(F - f(t)). \tag{24}$$

Illustrations

To illustrate the application of the NSRL model, time-series data for four medical innovations were examined. These innovations include ultrasound, CT head scanner, CT body scanner, and mammography. The data, however, are not census data. They were collected from a survey of 206 hospitals throughout the United States.[2] The hospitals were asked to identify themselves as adopters or nonadopters. If adopters, they provided the date of adoption (see [17] for details of the survey). The survey indicated that by 1978, of the 206 hospitals, 168 (81.55%) had adopted ultrasound, 113 hospitals (54.85%) had adopted the CT head scanner, 97 hospitals (47.09%) had adopted the CT body scanner, and 119 hospitals (57.77%) had adopted mammography.

In order to evaluate the effectiveness of NSRL, parameter estimates for the Blackman/Fisher-Pry model and the Sharif-Kabir model were also generated. All these models were fitted using a nonlinear programming algorithm. Table 1

TABLE 1
Parameter Estimates for Medical Technological Innovations

	Blackman/Fisher-Pry		Sharif-Kabir			NSRL		
Product	b	F	b	F	σ	b	F	δ
Ultrasound	0.4937	0.92	0.4936	0.92	0.0	0.7535	0.80	1.2191
CT head scanner	2.1518	0.39	2.4120	0.55	1.0	0.9645	0.56	0.6644
CT body scanner	2.2531	0.39	2.6054	0.52	1.0	1.3996	0.47	0.7899
Mammography	0.6722	0.61	0.6722	0.61	0.0	0.8735	0.56	1.1215

[2]For a discussion of how the estimates based on the survey type adoption data can be used for the adopter population, see Schmittlein and Mahajan [18].

TABLE 2
Points of Inflection

Product	Blackman/Fisher-Pry	Sharif-Kabir	NSRL
Ultrasound	$0.5F$	$0.5F$	$0.55F$
CT head scanner	$0.5F$	$0.33F$	$0.40F$
CT body scanner	$0.5F$	$0.33F$	$0.44F$
Mammography	$0.5F$	$0.5F$	$0.53F$

TABLE 3
Fit Statistics for Medical Technological Innovations

	Mean absolute deviation			Mean squared error		
Product	Blackman/ Fisher-Pry	Sharif-Kabir	NSRL	Blackman/ Fisher-Pry	Sharif-Kabir	NSRL
Ultrasound	2.73	2.73	2.00	12.42	12.42	6.28
CT head scanner	5.61	4.58	2.78	55.27	36.74	10.48
CT body scanner	3.53	2.32	1.75	23.66	7.69	4.08
Mammography	1.67	1.67	1.33	5.23	5.23	3.13

summarizes the parameter estimates. Points of inflection are shown in Table 2. The mean absolute deviation and mean squared error are reported in Table 3, and the actual and fitted market share estimates for the four products are depicted in Figures 2 through 5. The time-varying estimates of the coefficient of imitation are plotted in Figure 6. Some important comments on these results are warranted:

For ultrasound and mammography, the fitted values of the nonuniform influence factor δ of the NSRL model are greater than 1.0, indicating that influence is increasing with penetration. The Sharif-Kabir model does not allow increasing influence and so the Sharif-Kabir fitted values of σ for both these innovations are 0.0, indicating a constant coefficient of imitation. The Sharif-Kabir model reduces to the Blackman/Fisher-Pry model.

The values of the nonuniform influence factor in the NSRL model are less than 1.0

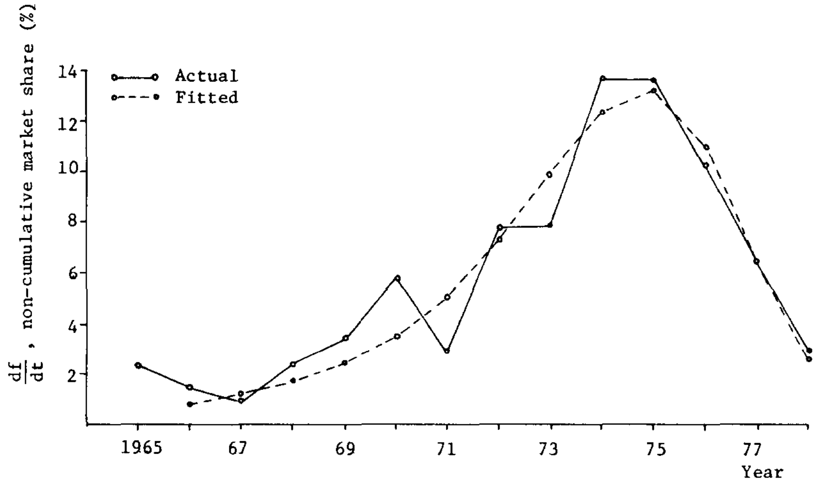

Fig. 2. Actual and NSRL fitted noncumulative share for ultrasound.

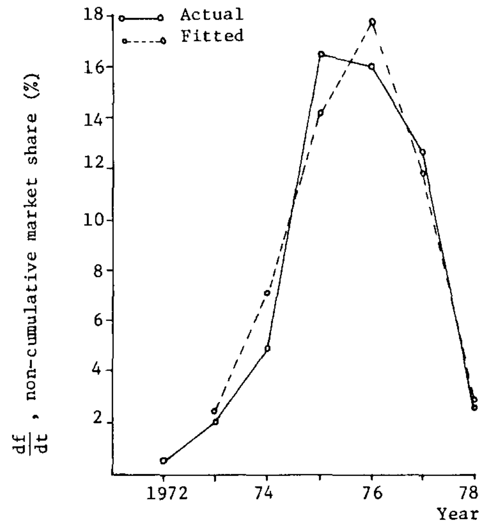

Fig. 3. Actual and NSRL fitted noncumulative market share for CT scanner head.

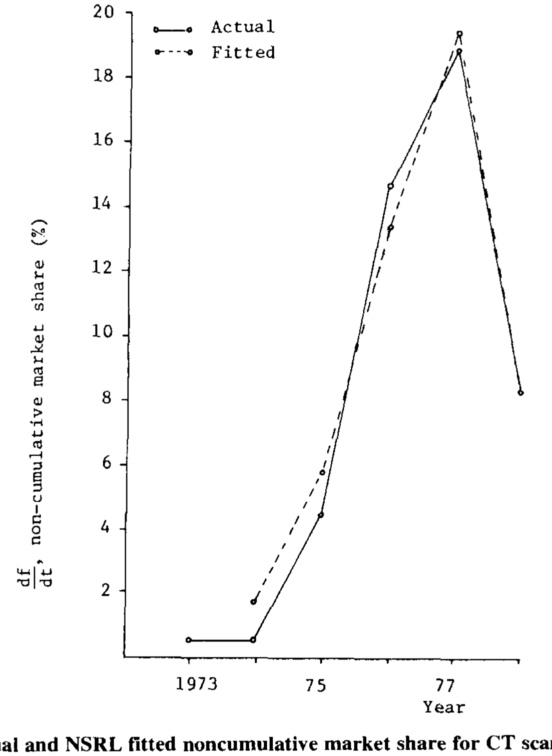

Fig. 4. Actual and NSRL fitted noncumulative market share for CT scanner body.

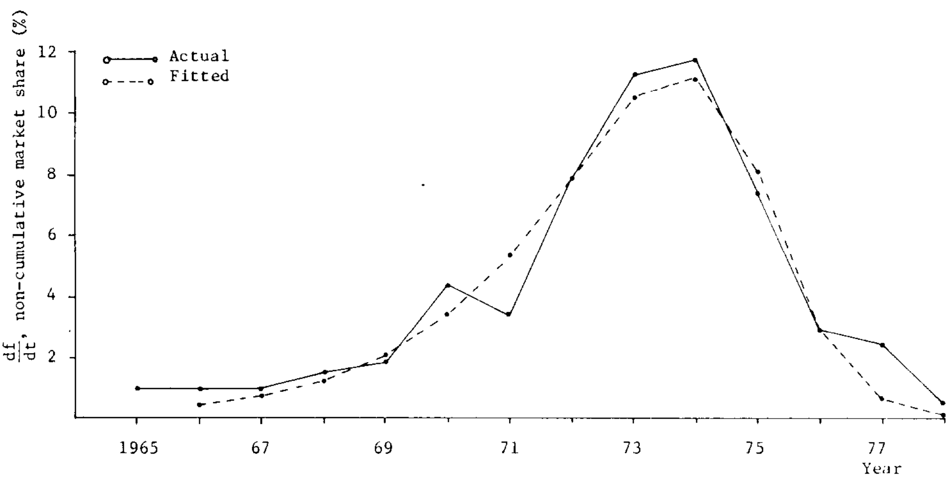

Fig. 5. Actual and NSRL fitted noncumulative market share for mammography.

for two innovations, the CT head scanner and the CT body scanner. This indicates high initial influence decreasing with penetration. For these two innovations, σ in the Sharif-Kabir model equals 1.0. The Sharif-Kabir model reduces to the Floyd model.

Table 3 indicates that the NSRL model provides a better fit for each of the four innovations than both of the other models. On average, the mean absolute deviation is 72% and 44% higher for the Blackman/Fisher-Pry and Sharif-Kabir models, respectively, than the NSRL model. The average mean squared errors are 303% and 159% higher, respectively. For the two innovations with high initial influence (indicated by $\delta < 1.0$) the Sharif-Kabir model that, in these two cases, reduces to the Floyd model, provides a useful improvement over the Blackman/Fisher-Pry model while not approaching the NSRL model.

Figure 6 shows that the NSRL estimates of the coefficient of imitation vary considerably. For the two scanner machines, the final period values are 21% and 38% of the initial period values. For the innovations with increasing values of the coefficient of imitation, ultrasound and mammography, final values are 2.17 and 1.66 times starting values, respectively. The Sharif-Kabir estimates decline to or close to zero for the scanners and are constant for ultrasound and mammography.

In order to compare the predictive validity of the models, one-step-ahead forecasts were generated for the final four years (see Table 4). Short time series meant that the scanner machines were excluded. For the remaining innovations, ultrasound and mammography, the Sharif-Kabir model reduces to the Blackman/Fisher-Pry model. The average values of the mean absolute deviation and the mean squared error for Blackman/Fisher-Pry/Sharif-Kabir were 97% and 299% higher than the NSRL values.

Thus, for these innovations the NSRL models fits the data very well and by means of the flexibility allowed by the nonuniform influence factor δ allows the model to respond to the substitution process.

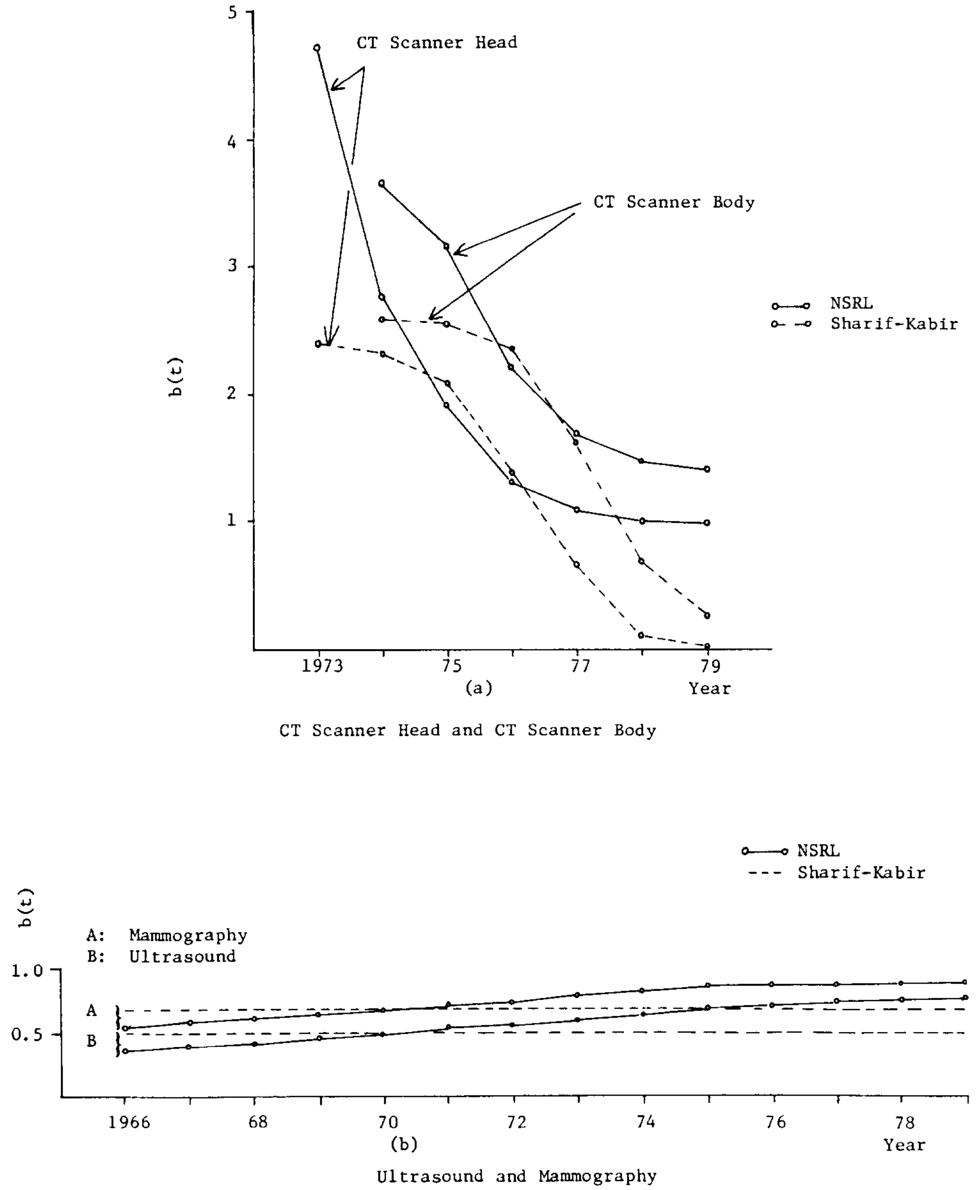

Fig. 6. Time-varying values of the coefficient of imitation for Sharif-Kabir and NSRL models.

Conclusions

A number of other models have been proposed to represent the time-dependent aspects of the substitution process. Stapleton [21] has suggested the utilization of the normal distribution to generate the substitution curves. However, the cumulative normal distribution also yields a S-shaped curve with a fixed point of inflection occurring at $f = F/2$. Related to the equation (1) representation of the substitution curve is the Gompertz curve, which has also been used to model the substitution process [15]. Although

TABLE 4
One-Step-Ahead Forecast Performance

		Mean absolute deviation		Mean squared error	
Product	Forecast period	Blackman/Fisher-Pry/ Sharif-Kabir	NSRL	Blackman/Fisher-Pry/ Sharif-Kabir	NSRL
Ultrasound	1975–78	7.6	3.7	67.3	18.3
Mammography	1975–78	4.4	2.4	36.8	7.9

this curve is nonsymmetric, it possesses a fixed point of inflection occurring at $f = e^{-1}F = 0.37F$.[3]

Another popular model that has been proposed to study the growth of new products (consumer durables) is the Bass model [1]. In addition to assuming the diffusion by imitation, this model also considers adoption by innovators who are not influenced by the current adopters. As discussed in detail in Easingwood et al [5]. This model is also symmetric.

This article has suggested a simple model of technological substitution termed as nonsymmetric responding logistic. Based on the theory that substitution is an imitation process, the model can accommodate different patterns of technological substitution. It allows the coefficient of imitation to vary over time systematically, the substitution curve to be symmetrical as well as nonsymmetrical, and the point of inflection to be responsive to the substitution process. Further work is currently underway to examine the application of NSRL for different types of innovations and to develop a taxonomy of substitution curves based on the nonuniform influence factor δ and the coefficient of imitation. These developments hopefully would lead to a procedure to predict the growth of a new product or technology before it is launched so that the go/no-go decisions in the light of the various launching strategies can be evaluated.

Appendix A: Sharif and Kabir/Floyd Models

Sharif and Kabir [20] have suggested that a smooth S-shaped curve within the region bounded by the Blackman/Fisher-Pry and Floyd curves can be obtained by combining these two models. That is,

$$\ln \frac{f}{F-f} + \frac{\sigma F}{F-f} = c_0 + ct \tag{A-1}$$

where $0 \leq \sigma \leq 1$, c_0 and c are constants, and $\sigma = 1$ yields the model suggested by Floyd [7]. Differentiation of equation (A-1) with respect to time t gives the following results:

$$\frac{d}{dt}\left[\ln f - \ln(F-f) + \frac{\sigma F}{F-f}\right] = c$$

or

$$\left[\frac{1}{f} + \frac{1}{F-f} + \frac{\sigma F}{(F-f)^2}\right]\frac{df}{dt} = c$$

[3]A modification of the Gompertz curve to accommodate the point of inflection that responds to the substitution process is discussed in Easingwood et al. [5].

or

$$\left[\frac{F}{f(F-f)} + \frac{\sigma F}{(F-f)^2}\right]\frac{df}{dt} = c$$

or

$$\left(\frac{(F-f+\sigma f)}{f(F-f)^2}\right)\frac{df}{dt} = \frac{c}{F}$$

or

$$\frac{df}{dt} = \frac{c}{F} \cdot \frac{f(F-f)^2}{F-f(1-\sigma)}$$

or

$$\frac{df}{dt} = \left[\frac{c(F-f)}{F(F-f(1-\sigma))}\right] f(F-f). \qquad \text{(A-2)}$$

Equation (A-2) represents the differential equation form for the Sharif and Kabir model. The Floyd model can be obtained by substituting $\sigma = 1$ into this equation. That is,

$$\frac{df}{dt} = \left[\frac{c(F-f)}{F^2}\right] f(F-f). \qquad \text{(A-3)}$$

Appendix B: Point of Inflection for Sharif and Kabir/Floyd Models

As derived in Appendix A, the Sharif and Kabir model is:

$$\frac{df}{dt} = \left(\frac{c(F-f)}{F(F-f(1-\sigma))}\right) f(F-f). \qquad \text{(B-1)}$$

The point of inflection can be obtained by differentiating equation (B-1) with respect to f and solving the resulting equation for f by equating it to zero. That is,

$$\frac{d}{df}\left(\frac{c}{F} \cdot \frac{f(F-f)^2}{F-f(1-\sigma)}\right) = 0$$

or

$$\frac{c}{F}\left[(F-f(1-\sigma))\left((F-f)^2 - 2f(F-f)\right) + f(F-f)^2(1-\sigma)\right] = 0$$

or

$$(F-f(1-\sigma))(F-3f) + f(F-f)(1-\sigma) = 0. \qquad \text{(B-2)}$$

Simplification of equation (B-2) gives

$$f^2(2-2\sigma) - 3fF + F^2 = 0. \qquad \text{(B-3)}$$

When $\sigma = 1$, equation (B-3) yields $f = F/3$, which is the point of inflection for the Floyd model. In general, the solution of equation (B-3) is given by

$$f = \frac{3F - \sqrt{F^2 + 8\sigma F^2}}{4(1-\sigma)}. \qquad \text{(B-4)}$$

When $\sigma = 0$, equation (B-4) yields $f = F/2$, which is the point of inflection for the Blackman/Fisher-Pry model.

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Received 7 August 1981