The Role of Within-Brand and Cross-Brand Communications in Competitive Growth

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Web Appendix A: The Solution of Equation 1

For the two-brand case, (i = 1, 2) the growth equations of our model are given by:

$$\text{(WA1)} \qquad \frac{dN_i(t)}{dt} = \left(p_i + \frac{q_i N_i(t)}{m} + \frac{q_{ij} N_j(t)}{m} \right)(m - N_i(t) - N_j(t))$$

$N_i(t)$ - Number of subscribers of the focal brand i at time t

$N_j(t)$ - Number of subscribers of the other brand j at time t

$N(t)$ - Total number of adopters ($N = N_i + N_j$)

m - Common market potential

p_i - Parameter of external influence for brand i

q_i - Within-brand word of mouth influence on brand i

q_{ij} - Cross-brand word of mouth influence of brand j on brand i

Without loss of generality, we assume that Firm i was first to the market, and Firm j joined at time t_0, therefore the initial conditions are $N_1(t_0) = N_0$, $N_2(t_0) = 0$.
In the case of similar brands, we assume that $q_1 = q_2 = q$ and $q_{12} = q_{21} \equiv q_{ij}$. We will later apply also an equality of the parameters p. Thus, for $t>t_0$ Equation WA1 becomes:

$$\text{(WA2)} \quad \frac{dN_i(t)}{dt} = \left(p_i + \frac{qN_i(t)}{m} + \frac{q_{ij} N_j(t)}{m} \right)(m - N(t))$$

Since $N(t)=N_1(t)+N_2(t)$ the Equation WA2 for each brand can be expressed as a function of its penetration $N_i(t)$ and the total category adopters $N(t)$, to get:

$$\text{(WA3)} \quad \frac{dN_i(t)}{dt} = \left(p_i + \frac{qN_i(t)}{m} + \frac{q_{ij}(N(t) - N_i(t))}{m} \right)(m - N(t))$$

Note, that $N(t)$ is known - this is the solution to the category equation obtained by summing up the Equations (WA2) for both firms:

(WA4) $$\frac{dN(t)}{dt}=\left((p_i+p_j)+\frac{(q+q_{ij})N(t)}{m}\right)(m-N(t))$$

Equation WA4 is a standard category level diffusion Equation, with initial conditions $N(t_0)=N_0$. For simplicity of notations we set the origin of the time axis t_0 to be zero.
The solution of the diffusion equation in such conditions is given by (see Muller, Peres and Mahajan 2007):

(WA5) $$N(t)=\frac{m(S-e^{-(P+Q)t})}{S+\frac{Q}{P}e^{-(P+Q)t}}$$

Where $P=p_1+p_2$, $Q=q+q_{ij}$, and $S=\frac{m+(Q/P)N_0}{m-N_0}$.

Based on that, we can go back to Equation WA3 and re-write it as:

(WA6) $$\frac{dN_i(t)}{dt}=\left(p_i m-p_i N(t)+q_{ij}N(t)-q_{ij}N^2(t)\right)-(q_{ij}-q+qN(t)/m-q_{ij}N(t)/m)\cdot N_i(t)$$

Where $N(t)$ is given by Equation WA5.

Equation WA6 is a first order linear differential Equation in $N_i(t)$, where the terms in each brackets are known functions of t. Any equation of the form $dN_i(t)/dt=A(t)-B(t)N_i(t)$, where A and B are known functions of t can be solved by finding an integrating factor, which is a function $M(t)$ that satisfies $B(t)M(t)=dM(t)/dt$. Multiplying both wings by $M(t)$ we get $M(t)\frac{dN_i(t)}{dt}+\frac{dM(t)}{dt}N_i(t)=A(t)M(t)$, hence $\frac{d(N_i(t)\cdot M(t))}{dt}=A(t)M(t)$. $N_i(t)$ is reached through the integration of both wings:

(WA7) $$N_i(t)=\frac{1}{M(t)}\left(\int A(t)\cdot M(t)dt+const\right)$$

Thus $N_i(t)$ can be expressed using $A(t)$, which is a known function of t, and the integrating factor $M(t)$.
In order to find $M(t)$, we use its definition, $dM(t)/dt=B(t)M(t)$, which is equivalent to $\frac{dM(t)/dt}{M(t)}=B(t)$ and thus $\frac{d(\ln M(t))}{dt}=B(t)$. After integration we obtain:

(WA8) $M(t) = e^{\int B(t)dt}$

Now we can go back to Equation WA6 and solve it. In our case $A(t) = \left(p_i m - p_i N(t) + q_{ij} N(t) - q_{ij} N^2(t)\right)$, $B(t) = (q_{ij} - q + qN(t)/m - q_{ij} N(t)/m)$, and *N(t)* is given by Equation WA5. Replacing these three terms in Equations WA8 and WA7, and using the above initial conditions, leads (through a tedious yet straightforward integration), to:

(WA9a) $$N_1(t) = \frac{m(K_1 - L_1 e^{-(P+Q)t})}{S + \frac{Q}{P} e^{-(P+Q)t}} + \frac{m\left(S + \frac{Q}{P}\right)^{\alpha-1}(K-L)}{\left(S + \frac{Q}{P} e^{-(P+Q)t}\right)^{\alpha}}$$

(WA9b) $$N_2(t) = \frac{m(K - L e^{-(P+Q)t})}{S + \frac{Q}{P} e^{-(P+Q)t}} - \frac{m\left(S + \frac{Q}{P}\right)^{\alpha-1}(K-L)}{\left(S + \frac{Q}{P} e^{-(P+Q)t}\right)^{\alpha}}$$

Where $P=p_1+p_2$; $Q=q+q_{ij}$; $S = \frac{m + (Q/P)N_0}{m - N_0}$; $\alpha = \frac{q - q_{ij}}{q + q_{ij}}$; $K = \frac{S \cdot (p_1 - p_2 + q - q_{ij})}{2(q - q_{ij})}$;

$L = \frac{p_2 q - p_1 q_{ij}}{(q - q_{ij})(p_1 + p_2)}$; $K_1 = \frac{S \cdot (p_2 - p_1 + q - q_{ij})}{2(q - q_{ij})}$; $L_1 = \frac{p_1 q - p_2 q_{ij}}{(q - q_{ij})(p_1 + p_2)}$.

If the products are similar so that the external influence parameter p_1, p_2 are equal $p_1=p_2$, Equation WA9 is reduced to:

(WA10) $$N_i(t) = (m/2)\frac{S - e^{-(P+Q)t}}{S + (Q/P)e^{-(P+Q)t}} + (-1)^{i+1} m \frac{\left(S + Q/P\right)^{\alpha-1}(S-1)/2}{\left(S + (Q/P)e^{-(P+Q)t}\right)^{\alpha}}$$

Where $S = \frac{m + (Q/P)N_0}{m - N_0}$; $\alpha = \frac{q - q_{ij}}{q + q_{ij}}$; i=1,2.

Note that the first term of Equation WA10 is the Bass function with non-zero initial condition. The parameter *S* represents the influence of the additional initial seed of adopters, and was termed the ***seeding factor*** by Muller, Peres, and Mahajan (2007). The second part of Equation WA10 is added to the first entrant and subtracted from the follower, representing the asymmetry in initial conditions. Summing up Equation set WA10 for i, j = 1,2, yields the Bass equation with non-zero initial conditions.

Web Appendix B: The Conditions Under Which the Follower Becomes the Market Leader

We discuss here the conditions under which the follower becomes the market leader. We have proven that if the growth parameters are equal (Equation 2), then, for any value of q_{ij}, the pioneer will maintain its advantage. In the case where the growth parameters *p* and *q* of the follower are larger than those of the pioneer (which is consistent with Equation 1), the follower might take over. The goal of this analysis is to examine how large should *p* and *q* of the follower be, in order to break the pioneering advantage. If we use the metaphor comparing the interaction advantage to a current that drags the late entrant, then the follower has to row against the current – we want to measure how strong the current is.

Method

The analysis is done using a full factorial experiment on the model parameters based on the growth function in Equation 1. For each parameter combination we measured by how much should *p* and *q* of the follower be multiplied in order to pass the pioneer. The simulations were done using Mathematica, and the algorithm was finding the minimum multiplier which will enable making the pioneer first at a certain time point. We heuristically set this time point to be 30. Note, that this is a conservative choice, if we wanted the follower to become the leader earlier, multipliers should have been higher.

The equations are:

$$\frac{dN_i}{dt} = (p + \frac{qN_i}{m} + \frac{q_{ij}N_j}{m})(m - N_i - N_j)$$
$$\frac{dN_j}{dt} = (\alpha \quad p + \frac{\alpha \quad qN_j}{m} + \frac{q_{ij}N_i}{m})(m - N_i - N_j)$$

Where α is the multiplier, and firms *i* and *i* are the pioneer and late entrant, respectively. The follower enters at time t_0.

The full factorial combination was performed with the following parameter values:

p: 0.005 to 0.1 in steps of 0.005
q: 0.1 to 0.5 in steps of 0.1
q_{ij}: 0.1 to q in steps of 0.1
t_0: 2 to 10 in steps of 1

The parameter range was set to crease reasonable values, so that $p < q$ and p , $q < 1$ before and after the multiplication by α. Altogether we got 2700 combinations. The results are that the “current” is strong – in most cases the resulting multiplier caused αp or αq to be larger than 1. When excluded those cases, the average multiplier is 2.14