

TRIAL/AWARENESS ADVERTISING DECISIONS

A Control Problem with Phase Diagrams with Non-Stationary Boundaries

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The paper presents a solution to a diffusion process in which advertising has two goals – Awareness Creation and Trial Inducement. It is shown that the advertising time paths can be either monotonic or single peaked. The solution technique involves phase diagrams where the boundaries are non-stationary. General discussion of such techniques is provided. The paper proves the existence of a stationary equilibrium to which the paths converge. Furthermore, this two state variables, two control variables equilibrium point is conditionally stable.

1. Introduction

Most OR and Economics models depicting advertising in a dynamic context regard advertising as a single variable which directly or indirectly effects the revenues of the firm. See, for example, Gould (1970), Nerlove and Arrow (1962), Sethi (1977, 1979). Marketers, however, do not regard advertising as a single variable with a single objective. The objectives are typically stated in terms of a communications effect. The objectives will be stated in terms of *increasing awareness, changing attitude, changing predisposition to buy*, or some combination of the three. [See, for example, Tull and Hawkins (1980, p. 646) or any other marketing research textbook.]

This paper presents a dynamic model based on a diffusion process which makes the distinction between two types of advertising objectives — increasing awareness and changing predisposition to buy. For simplicity these types will be denoted throughout the paper as awareness versus trial advertising.

The paper investigates the optimal policy implication of such a model. The model is based on the Dodson and Muller model (1978). The latter is a diffusion model which generalizes several models both in economics and in management science, such as Gould (1970), Nerlove and Arrow (1962), Vidale and Wolfe (1957), Palda (1965), Bass (1969), Nicosia (1966) and

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Glaister (1974). The interested reader is referred to the survey papers by Sethi (1977) and by Mahajan and Muller (1979).

The model deals with an introduction of a new product in a market of size N , where N is divided into $x(t)$ – the number of people who are unaware of the existence of the product, $y(t)$ – the number of potential customers who are aware of the product but have not yet purchased it, and $z(t)$ – the number of current customers who have purchased the product. Since x , y , and z partition N into three distinct groups of ignorant, potential and current customers, it follows that

$$x(t) + y(t) + z(t) = N. \quad (1)$$

Advertising is broken down to its two components: (1) awareness (denoted by u) which informs consumers about the product and thus transfers them from the unaware group x into the potential group y , and (2) trial advertising (denoted by v) which persuades consumers to purchase the product and transfers them from the potential consumers group y into the current customers group z .

The flows of consumers from and into the different groups are given in the following transition equations:

$$\dot{x} = -ux - kx(N-x)/N, \quad x(0) = N, \quad (2)$$

$$\dot{y} = ux + kx(N-x)/N - (a+v)y + \delta z, \quad y(0) = 0, \quad (3)$$

$$\dot{z} = (a+v)y - \delta z, \quad z(0) = 0. \quad (4)$$

The sales rate is given by

$$s(t) = (a+v)y(t) + gz(t), \quad (5)$$

where k is the contact rate, a is the trial (first purchase) rate, g is the repeat purchase rate, and δ is the switching rate, i.e., the rate at which current customers are purchasing rival brands.

The explanation of eq. (2) is as follows: The people who know, $N-x$, contact and inform a total of $k(N-x)$, out of which only a fraction of x/N are newly informed. In addition, out of the total number of people informed via advertising uN , only a fraction of x/N are newly informed. Eqs. (3) and (4) are similarly constructed.

The advertising expenditures needed to have the awareness effect u and the trial effect v are denoted by $U(u)$ and $V(v)$, respectively. Those are assumed to be convex functions.

The present value of the profit stream to be maximized is given by

$$\int_0^{\infty} e^{-rt} \{ps(t) - U(u(t)) - V(v(t))\} dt,$$

where r is the discount rate, and p is the net price. It will be convenient to express the objective function as a function of one state variable z instead of the sales which involve three variables, y , v , and z .

Substituting from (5) for $s(t)$ and from (4) for the terms $(a+v)y(t)$ yields

$$\int_0^{\infty} e^{-rt} \{p[z + \delta z + gz] - U(u) - V(v)\} dt.$$

Integrating the term $e^{-rt}z$ by parts and using the initial condition $z(0)=0$ yields

$$\int_0^{\infty} e^{-rt} [cz - U(u) - V(v)] dt, \quad (6)$$

where c is given by

$$c = p(r + \delta + g). \quad (7)$$

The three equations (2)–(4) are dependent as they sum up zero. The choice of which two of the three to use is one of convenience. Eqs. (2) and (4) were chosen where the term $N - x - z$ was substituted for y , using (1). In the next section the maximization problem will be formally presented and solved. Analysis and discussion of the solution will follow.

2. Discussion of the necessary conditions

In this section the maximization problem of the firm facing the environment described by the diffusion process is presented and solved. Analysis and discussion of the solution will then follow.

The problem is to choose $u(t)$ and $v(t)$ such as to maximize eq. (6),

$$\int_0^{\infty} e^{-rt} [cz - U(u) - V(v)] dt,$$

subject to eqs. (2) and (4),

$$\dot{x} = -ux - kx(N - x)/N, \quad x(0) = N,$$

and

$$\dot{z} = (a + v)(N - x - z) - \delta z, \quad z(0) = 0,$$

for $u \geq 0$ and $v \geq 0$. The functions U and V satisfy $U', V' > 0$ and $U'', V'' > 0$ for all $u > 0$ and $v > 0$. $U'(0) = V'(0) = 0$. For example, quadratic functions will satisfy the above conditions. It is rather straightforward to check that the equations (2)–(4) and the initial conditions guarantee that both $x(t)$ and $z(t)$ are indeed bounded between zero and N . As for the equilibrium itself, it is possible to show its existence (see appendix 1) and its stability (see appendix 2).

What will follow now will be a formal solution of the problem. An interesting technical aspect of this problem is that it involves phase diagrams with non-stationary boundaries. For the pioneering work on this subject, see Kamien and Schwartz (1977); for the use of this technique for differential games, see Fershtman and Muller (1983).

Appending (2) with a multiplier $-\lambda(t)$ (so that λ would be non-negative) and (4) with a multiplier $+\mu(t)$ to the integrand yields a current value Hamiltonian H ,

$$H = cz - U(u) - V(v) + \lambda ux + \lambda kx(N - x)/N + \mu(a + v)(N - x - z) - \mu\delta z.$$

Since the variable y plays an important role, define y to be $y = N - x - z$, or using eqs. (2) and (4), one can generate eq. (3),

$$\dot{y} = ux + kx(N - x)/N - (a + v)y + \delta z, \quad y(0) = 0.$$

The state variables x, z , multiplier functions λ, μ , and control variables u, v have to satisfy not only (2) and (4) but also the following conditions if they are to maximize (6):

$$\partial H / \partial u = -U'(u) + \lambda x \leq 0, \quad u \partial H / \partial u = 0, \quad (8)$$

$$\partial H / \partial v = -V'(v) + \mu y \leq 0, \quad v \partial H / \partial v = 0, \quad (9)$$

and

$$\dot{\lambda} = r\lambda + \partial H / \partial x \quad \text{and} \quad \dot{\mu} = r\mu - \partial H / \partial z,$$

that is

$$\dot{\lambda} = (r + u + k(N - 2x)/N)\lambda - \mu(a + v), \quad (10)$$

$$\dot{\mu} = (r + a + v + \delta)\mu - c. \quad (11)$$

The objective now is to manipulate the above necessary conditions so as to be able to trace the behavior of the optimal path in the u, v space. It should be noted that altogether we have four time-autonomous differential equations with four unknowns [eqs. (2), (4), (12) and (14) and the variables x, z (or x and y), u and v]. Since the method of investigation is graphical, it involves the two-dimensional u, v space, thus each level of the state variables yields a different cross-sectional cut. Thus in the u, v space, the $\dot{u}=0$ and $\dot{v}=0$ boundaries will not be stationary as long as the state variables are not stationary.

If $\lambda(0) < 0$, then, according to (8), $u(0)=0$ and the optimal behavior for the firm is not to enter the market. If $\lambda(0) > 0$, then $u(0) > 0$, and as long as $u(t) > 0$, an equality holds in (8), that is

$$U'(u) = \lambda x. \quad (8')$$

If $\mu(0) < 0$, then according to (11), $\dot{\mu}(0) < 0$, and therefore $\dot{\mu}(t) < \dot{\mu}(0) < 0$ for all t , which rules out the convergence to a steady state. Therefore $\mu(0) > 0$. However, since $y(0)=0$, then according to (9), $v(0)=0$; otherwise $v\partial H/\partial v$ will not be zero. By the same argument, since $\dot{y}(0) > 0$, we have that for $t > 0$, $y(t) > 0$. That requires v to be positive since otherwise $\partial H/\partial v$ will be positive. The optimal policy then is to start with $v(0)=0$, but $v(t) > 0$ for all $t > 0$. The equality in (9) holds for all $t > 0$, i.e.,

$$V'(v) = \mu y. \quad (9')$$

Differentiating (8'), substituting (2) and (10) for \dot{x} and $\dot{\lambda}$, respectively, and substituting (8') for λ and (9') for μ yields

$$U''\dot{u} = \dot{\lambda}x + \lambda\dot{x} = (r + u + k(N - 2x)/N)\lambda x$$

$$- \mu(a + v)x - \lambda ux - \lambda kx(N - x)/N,$$

or

$$U''\dot{u} = U'(r - kx/N) - V'(a + v)x/y. \quad (12)$$

Thus in the u, v space, \dot{u} vanishes whenever (12') holds

$$U'(r - kx/N) = V'(a + v)x/y. \quad (12')$$

If $r > k$, this curve has a positive slope throughout. If $r < k$, then the line is not in the positive quadrant until x is small enough such that $r > kx/N$. In both cases, when the $\dot{u}=0$ locus appears in the positive quadrant of the u, v space, it has a positive slope. From above and to the left of $\dot{u}=0$, u is increasing.

Note that the appearance of x and x/y in (12) implies that this boundary is not stationary in the u, v space unless x and y are stationary.

Since $\dot{x} < 0$ throughout, $d(r - kx/N)/dt > 0$ for all t . If $d(x/y)/dt \leq 0$, then the $\dot{u} = 0$ boundary necessarily moves 'down' in the u, v space as depicted in fig. 1. Before turning to the $\dot{v} = 0$ locus, some discussion is needed in general on phase diagrams with non-stationary boundaries.

The discussion will rely on the derivation of the $\dot{u} = 0$ boundary in the u, v space (see fig. 1). The objective is to point out an argument which might arise in connection with the non-stationarity of one or more of the boundaries.

We first check the behavior of u at $t = 0$, by evaluating (12) when $x = N$ and $v = 0$,

$$U''\dot{u}|_{t=0} = N[(r - k)\lambda(0) - \mu(0)a]. \tag{13}$$

If $r > k$ and a is small enough (example $a = k = 0$), then $\dot{u}(0) > 0$. When this holds, it is evident from fig. 1 that the optimal path starts to the left of the $u = 0$ locus.

One might now argue that the path cannot cross the falling $\dot{u} = 0$ line since when it does, the path is stationary vertically (since $\dot{u} = 0$) but the $\dot{u} = 0$ locus

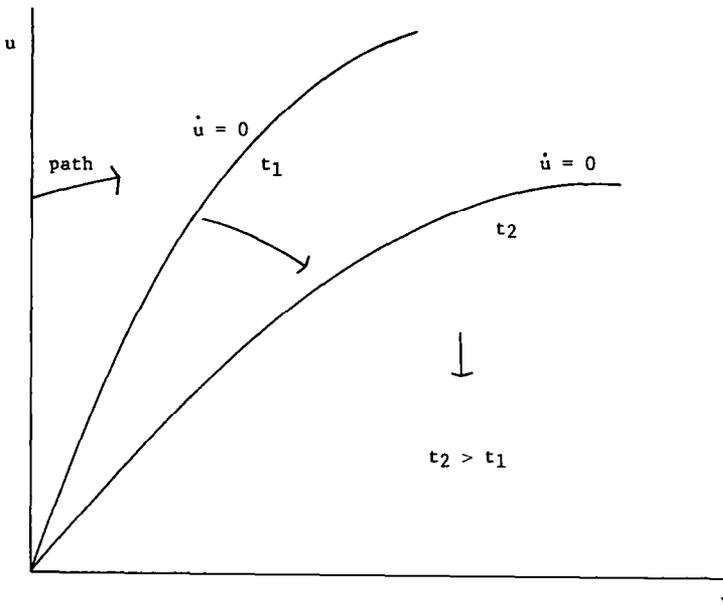


Fig. 1. The movement of the $\dot{u} = 0$ locus when $d(x/y)/dt \leq 0$. This movement will be referred to as a movement 'down'. The reverse movement will be called a movement 'up'. $\dot{u} = 0$ is $U'(r - kx/N) = V'(a + v)x/y$. The curved arrow denotes the movement of the $\dot{u} = 0$ locus. The two straight arrows imply that above and to the left of $\dot{u} = 0$, u is increasing while it is decreasing below and to the right of the $\dot{u} = 0$.

is not. Therefore, in the next instant the boundary would fall below the path, so that the path cannot intersect the boundary.

This argument will be referred to as the 'intersection argument'. This argument, despite its attractiveness, is incomplete and might be misleading. Since the validation of this last statement is rather complex technically, it is deferred to section 3. Its conclusion is that the discussion on the possible path is now rather complex since very little can now be utilized to rule out certain paths which seemingly satisfy the necessary conditions. Still some behaviors can be ruled out as will become evident shortly.

We turn now to the $\dot{v}=0$ locus. Differentiating (9'), substituting (3) and (11) for \dot{y} and $\dot{\mu}$, respectively, and substituting (9') for μ , yields

$$\begin{aligned} V''\dot{v} &= \dot{\mu} + \mu\dot{y} \\ &= (r + a + v + \delta)\mu y - cy + \mu ux + \mu kx(N-x)/N - \mu(a+v)y + \delta z\mu \\ &= rV' - cy + \mu ux + \mu xk(N-x)/N + \delta\mu(y+z), \\ &= V'[r + ux/y + xk(N-x)/Ny + \delta(N-x)/y] - cy, \end{aligned} \quad (14)$$

Thus the $\dot{v}=0$ boundary is

$$V'[r + (x/y)(u + k(N-x)/N + \delta(N-x)/x)] = cy, \quad (14')$$

This is clearly a decreasing curve in the u, v space. The intercept of this curve with the v axis is given by

$$V' = cy/[r + (x/y)(k(N-x)/N + \delta(N-x)/x)] > 0.$$

Since $\dot{x} < 0$ throughout, $d(k(N-x)/N + \delta(N-x)/x)/dt > 0$. In case that $d(x/y)/dt > 0$ and $\dot{y} < 0$, this line moves 'down', i.e., folds towards the origin as depicted in fig. 2. However, if $d(x/y)/dt < 0$ and $\dot{y} > 0$, this line may very well be moving in the opposite direction of the above movement, i.e., move 'up'.

From (14), above and to the right of $\dot{v}=0$, v is increasing, i.e., an increase either in μ or v will increase \dot{v} .

The next issue is where, in the u, v space, does the path start and where does it end. At $t=0$ from (12') and (14'), the $\dot{u}=0$ locus is the u axis, and the $\dot{v}=0$ is the v axis since $x(0)=N$ and $y(0)=0$.

The $\dot{v}=0$ locus starts immediately moving 'up' (see fig. 2 for the definition of moving up or down), while the $\dot{u}=0$ depends on the parameter configuration. To see precisely what happens to $\dot{u}(0)$, we use (12) with (8')

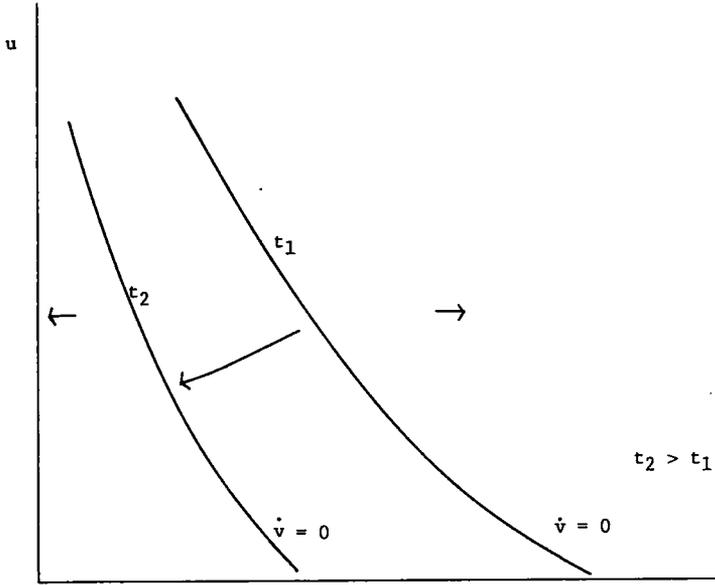


Fig. 2. The movement of the $\dot{v}=0$ locus when $\dot{y}<0$ and $d(x/y)/dt>0$. This movement will be referred to as a movement 'down'. The reverse movement will be called, obviously, 'up'. $\dot{v}=0$ is $V'[r+(x/y)(u+k(N-x)/N+\delta(N-x)/x)]=cy$. The curved arrow shows the movement of the $\dot{v}=0$ locus. The two straight arrows imply that above and to the right of $\dot{v}=0$, v is increasing, while below and to the left of $\dot{v}=0$, v is decreasing.

substituted for $U'(u)$ to get

$$U''\dot{u}|_{t=0} = \dot{\lambda}x + \lambda\dot{x}|_{t=0} = N((r-k)\lambda(0) - \mu(0)a).$$

If $r < k$, then $\dot{u}(0) < 0$. If $r > k$ and a is small enough (for example, $a = k = 0$), $\dot{u}(0) > 0$. A similar analysis reveals that $\dot{v}(0) > 0$. Thus the optimal path starts always above the $\dot{v}=0$ and on the $\dot{u}=0$. Immediately after $t=0$, it is either above or below the $\dot{u}=0$ depending on a , r , and k . This is shown in fig. 3. From eq. (2) $\dot{x}=0$ if and only if $x=0$. In this case u necessarily is zero since any positive u will yield the same result because $ux=0$. In contrast, the steady state level of v , denoted by v^* , is positive satisfying jointly with x , λ and μ , $\dot{z}=0$, $\dot{\lambda}=0$, $\dot{\mu}=0$, $V' = \mu(N-z)$.

The objective now is to rule out a path which moves counter-clockwise as is shown in fig. 5a.

In order to do that, we need to trace the movement of another locus, the $d(x/y)/dt=0$.

Upon using (2) and (3) the equation $d(x/y)/dt=0$ can be written as

$$(x/y + 1)(u + k(N-x)/N) + \delta(N/y - x/y - 1) = a + v. \tag{15}$$

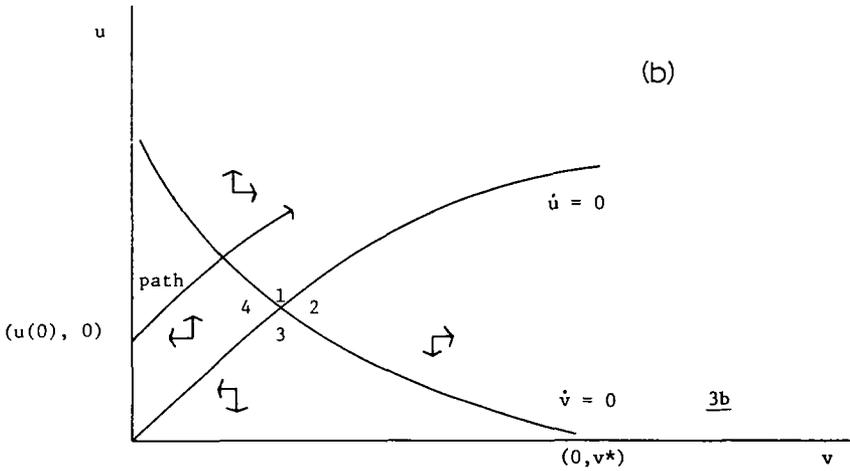
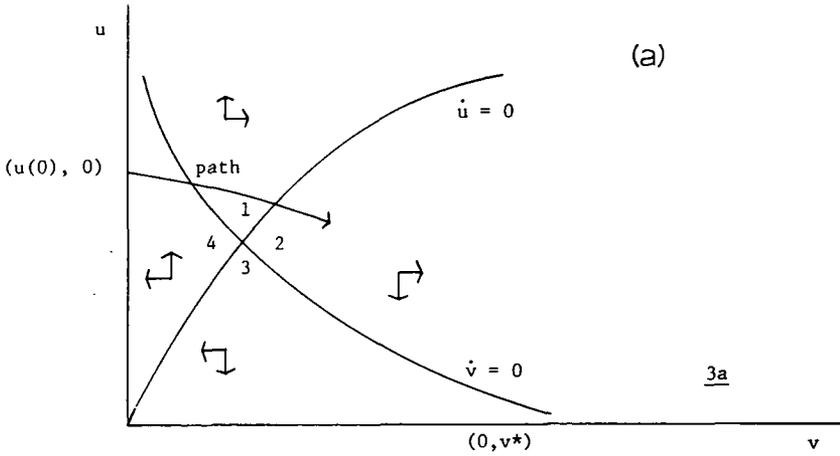


Fig. 3. At $t=0$, the $\dot{u}=0$ locus is the u axis, while the $\dot{v}=0$ locus is the v axis so that the path can initially be in region 1 or region 2 depending on the initial relative rate of movement (speed) of the loci and the path.

Suppose the path now is in region 1 of fig. 3. It might have begun there or might have begun in region 2 but then was caught by the $\dot{u}=0$ locus. In order for the $\dot{v}=0$ to catch up with the path, the path could not have already caught up with the $d(x/y)/dt=0$ locus. If it did and $d(x/y)/dt > 0$, this implies that $\dot{y} < 0$ (since $\dot{x} < 0$ for all t) so that the $\dot{v}=0$ moves down (see fig. 2). This movement is in the opposite direction of the movement of the path so that no intersection is possible. Therefore, the $\dot{v}=0$ locus caught up with the path while $d(x/y)/dt < 0$. This is shown in fig. 4.

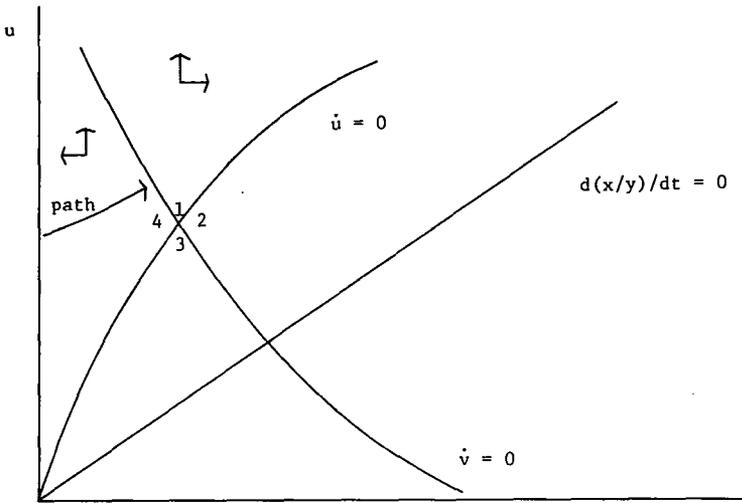


Fig. 4. $\dot{u}=0$ is $U'(r-kx/n)=V'(a+v)x/y$, $\dot{v}=0$ is $V'[r+(x/y)(u+k(N-x)/N+\delta(N-x)/x)]=cy$, $d(x/y)/dt=0$ is $(x/y+1)(u+k(N-x)/N)+\delta(N/y-x/y-1)=a+v$.

Now the path is in region 4. The $d(x/y)/dt=0$ locus cannot now catch the path, since when it does, from (15) it is clear that since $d(x/y)/dt=0$ (and so $\dot{y}<0$), the $d(x/y)/dt=0$ locus will be falling. Since the path is moving in a northwesterly direction, no intersection takes place. That implies that $\dot{u}=0$ locus will continue to move down since $d(x/y)/dt<0$. This means that the path cannot intersect the $\dot{u}=0$ locus. Since $\dot{u}>0$, it will imply that $\dot{u}>0$ for all t . This contradicts the steady state being composed of $v>0$ and $u=0$. Therefore, we conclude that if there exists a path leading to a steady state, it cannot move counter-clockwise. Such a path is shown in fig. 5b.

It should be noted that this is the general pattern possible. It still might have several extrema both in u and in v . Though these behaviors certainly are counter-intuitive, they cannot be ruled out rigorously without relying on the 'intersection' argument given before.

The temporal paths of awareness and trial advertising are given in fig. 6. In all cases the trial advertising gradually increases while the awareness advertising is either monotonic or single peaked. As the product matures, the awareness/trial ratio decreases (not necessarily monotonically). In the steady state itself, advertising is purely trial. Since the steady state is not attainable in finite time, advertising consists of a combination of trial and awareness advertising for all finite time.

The result of virtually all models [with the exception of Naslund (1979)] is to have a monotonically decreasing advertising over time. This, however, does not have to hold if several objectives of different types of advertising are sought. Thus, awareness advertising can indeed decline over time. However,

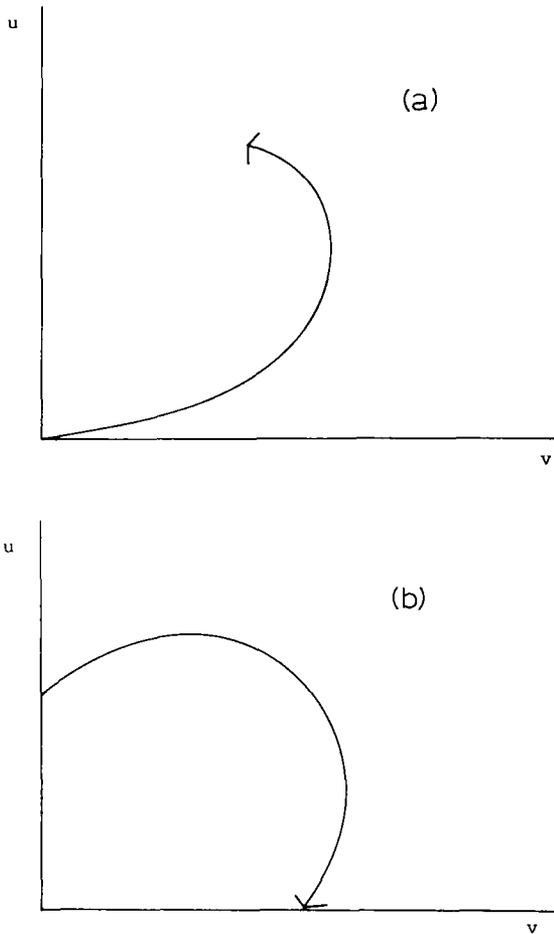


Fig. 5

since trial advertising will increase, the total expenditures will depend on the relative costs of having these respective effects. Specifically, the total advertising budget expenditure $E(t) = U(u) + V(v)$. If u is initially increasing, then this expenditure E increases with time. If u is decreasing initially and thus decreasing throughout, then \dot{E} can be negative or positive, depending on the relative magnitudes of the marginal costs U' and V' .

As mentioned before, one exception is the paper by Naslund which relies on the model by Nicosia. Naslund found some condition under which *pulsing* is an optimal policy where this policy calls for alternating periods of high and zero advertising levels. The problem with his analysis is that the conditions are rarely met. See appendix 3 for a formal proof.

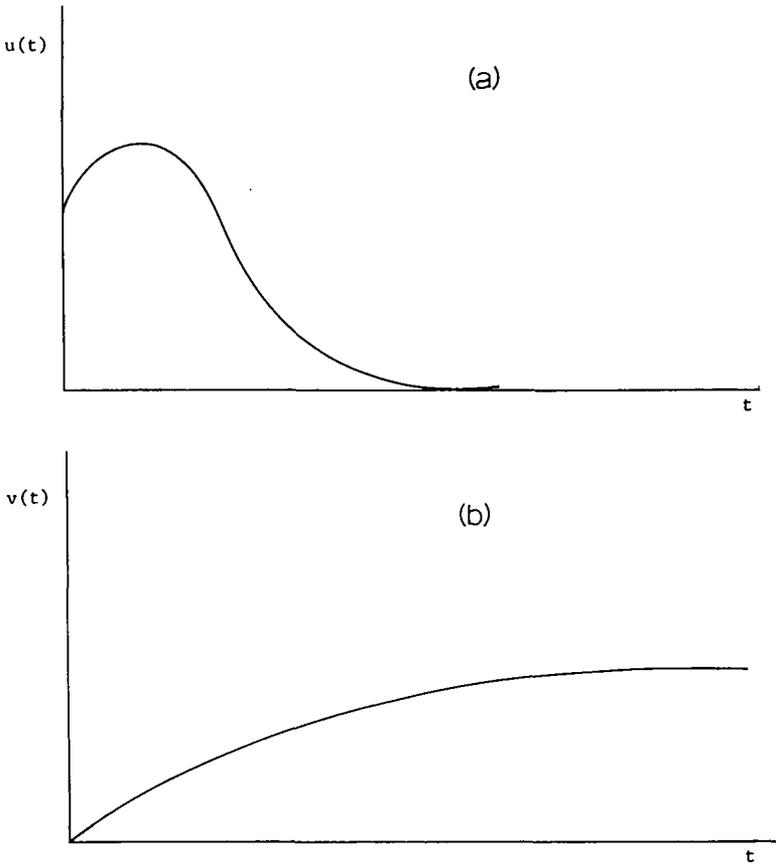


Fig. 6. The temporal paths of awareness and trial advertising.

3. The intersection argument

The discussion in this section will rely on section 2, particularly on the derivation of the $\dot{u}=0$ boundary of the u, v space (see fig. 1). The objective in this section is to rule out an argument which might arise in connection with the non-stationarity of one or more of the boundaries.

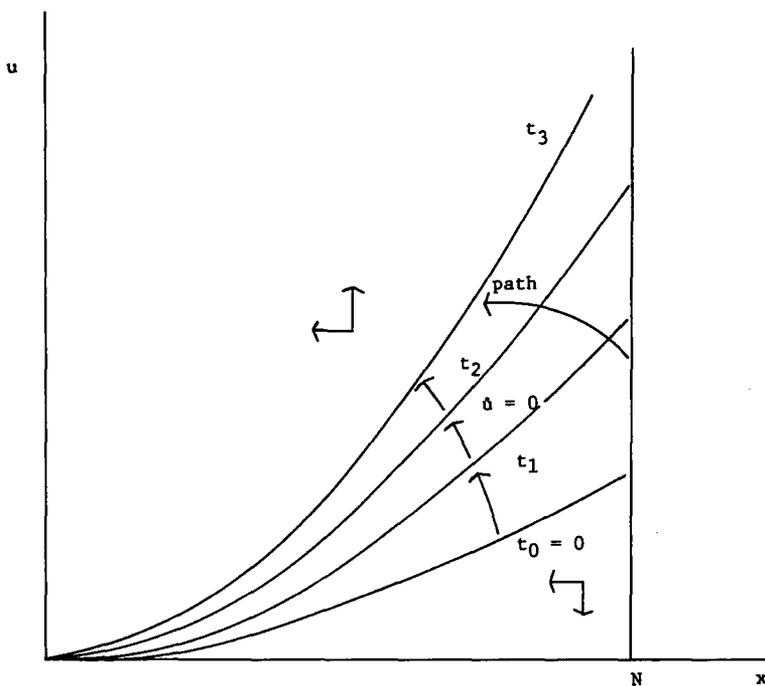
For the remainder of this section, assume that $\dot{u}(0) > 0$. From fig. 1, it is evident that the optimal path starts to the left of the $\dot{u}=0$ locus. One might now argue that the path cannot cross the falling $\dot{u}=0$ line since when it does, the path is stationary vertically (since $\dot{u}=0$) but the $\dot{u}=0$ locus is not. Therefore, in the next instant the boundary would fall below the path so that the path cannot intersect the boundary. The argument will be referred to as the 'intersection argument'. This argument, despite its attractiveness, might be misleading. To see that, we reconsider the same movements as projected

in the u, x space (fig. 7),

$$\dot{u}=0 \text{ is } U'(r-kx/N)=V'(a+v)x/y.$$

The $\dot{u}=0$ is moving 'up' if $\dot{v}>0$ and $\dot{y}<0$. The path has to move to the left of the boundary since $\dot{u}(0)>0$. We now employ the argument in the u, v space. As long as $\dot{y}>0$, the $\dot{u}=0$ locus moves down since $d(x/y)/dt<0$ (see fig. 1). Assume, therefore, that $\dot{y}<0$ but such that $d(x/y)/dt$ is still negative. This is clearly possible since $\dot{x}<0$. The $\dot{u}=0$ boundary in the u, v space is falling, thus at the moment of intersection of the path and the boundary, the path is stationary vertically while the locus falls. Therefore, the $\dot{u}=0$ locus falls below the path, and no crossing has been made.

Consider, at the same moment, the u, x space (fig. 7). Since $\dot{y}<0$, the path moves 'up'; therefore, at the moment of intersection, the path is stationary vertically ($\dot{u}=0$) while the $\dot{u}=0$ locus moves up. Therefore at the next



$$t_3 > t_2 > t_1 > t_0$$

Fig. 7. $\dot{u}=0$ is $U'(r-kx/N)=V'(a+v)x/y$ and $t_3>t_2>t_1>t_0$. The movement according to the curved arrows (called 'up') holds as long as $\dot{v}>0$, $\dot{y}<0$. The intersection occurred at $t=t_2$. Thus the path was above the $\dot{u}=0$ locus before t_2 , and below it thereafter. .

moment the locus will be above the path, thus the crossing has been established. Thus, relying only on one phase diagram might lead to the wrong conclusion.

The problem lies in the fact that the above argument takes into account the vertical movements of the locus and the path but fails to consider their horizontal movements.

In general, it is clear that the occurrence of an intersection depends on the horizontal movement of the 'path' (denoted by \dot{v}_p) versus the horizontal movement of the 'locus' (denoted by \dot{v}_e). In order for the path to intersect and cross the $\dot{u}=0$ locus, $\dot{v}_p > \dot{v}_e$. This condition can be expressed as follows:

Suppose that $\dot{u} = F(u, v, t)$. So the $\dot{u}=0$ locus is given by $F(u, v, t) = 0$. For definiteness, suppose that $F_u > 0$, $F_v < 0$ and $F_t > 0$, as illustrated in fig. 8.

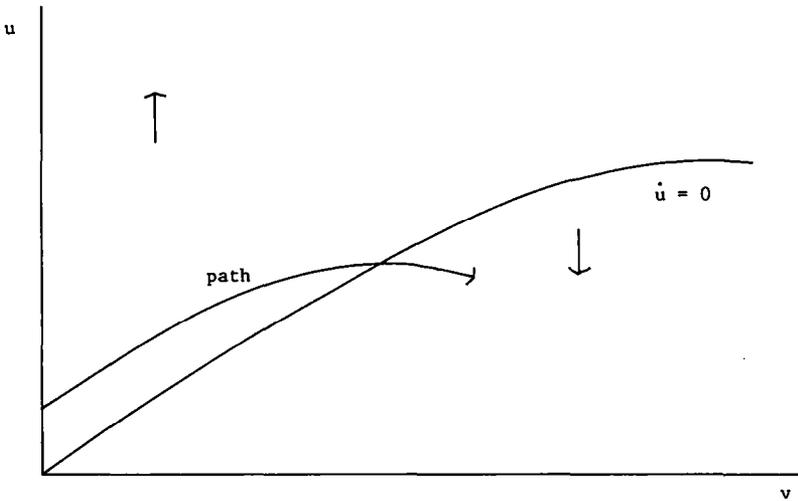


Fig. 8. $\dot{u} = F(u, v, t)$, so that the $\dot{u}=0$ locus is given by $F(u, v, t) = 0$, where $F_t > 0$, $F_u > 0$, $F_v < 0$.

\dot{v}_e is computed from differentiating the $\dot{u}=0$ locus with respect to time while keeping u fixed, i.e.,

$$d[F(u, v, t)]/du|_{\dot{u}=0} = F_u \dot{u} + f_v \dot{v} + F_t = F_v \dot{v}_e + F_t = 0,$$

or

$$\dot{v}_e = -F_t/F_v.$$

The condition for crossing $\dot{v}_p > \dot{v}_e$ can be written as

$$\dot{v}_p > F_t/F_v.$$

In principle, the application of this condition seems straightforward. F_t and F_v are calculated by differentiation of the $\dot{u}=0$ locus. \dot{v}_p (the horizontal movement of the path) is given by the necessary conditions since they have to be satisfied by the path.

Appendix 1: Existence of an equilibrium

Since at equilibrium, $u=0$, eq. (2) implies that $x=0$. Setting eqs. (4), (9), (10) and (11) equal to zero yields the following four-by-four system:

$$\delta z = (a+v)(N-z), \quad (\text{A.1})$$

$$V'(v) = \mu(N-z), \quad (\text{A.2})$$

$$(r+k)\lambda = \mu(a+v), \quad (\text{A.3})$$

$$(r+a+v+\delta)\mu = c. \quad (\text{A.4})$$

Eqs. (2) and (8) now become identities. Given μ and v eq. (A.3) determines λ . Thus we can regard eqs. (A.1), (A.2) and (A.4) as a three-by-three system. Substitute μ from (A.4) into (A.2), and substitute z from (A.1) into (A.2) to achieve one equation in v , i.e.,

$$(a+\delta+v)(r+a+v+\delta)V'(v) = cN\delta. \quad (\text{A.2}')$$

When $v=0$, the L.H.S. of (A.2') vanishes. When $v \rightarrow \infty$, L.H.S. $\rightarrow \infty$. Since it is continuous in v , then there exists a value v^* which satisfies (A.2'). The levels of μ^* and z^* will then be found from eqs. (A.4) and (A.1), respectively.

Appendix 2: Stability of the equilibrium

Substituting for y from eq. (1) yields four differential equations for the four variables in question: x , z , u and v [eqs. (2), (4), (12) and (14)]. Linearizing the equations in the standard way, i.e., expanding them to a Taylor series, around the equilibrium point x^* , z^* , u^* , v^* taking the linear part only, the homogeneous system becomes the following:

$$\dot{x} = -kx, \quad (\text{A.5})$$

$$\dot{u} = -\frac{V'(v^*)(a+v^*)}{(N-z^*)U''(u^*)}x + ru, \quad (\text{A.6})$$

$$\dot{z} = -(a + v^*)x - (a + v^* + \delta)z + (N - z^*)v, \quad (\text{A.7})$$

$$\begin{aligned} \dot{v} = & L^*[2c(N - z^*) - V'(v^*)(r + \delta - k)]x \\ & + L^*[2c(N - z^*) - rV'(v^*)]z + [r + \delta N/(N - z^*)]v, \end{aligned} \quad (\text{A.8})$$

where

$$L^* = 1/V''(v^*)(N - z^*).$$

Consider the following partitioned matrix:

$$\frac{\partial(\dot{x}, \dot{u}, \dot{z}, \dot{v})}{\partial(x, u, z, v)} = \left[\begin{array}{cc|cc} -k & 0 & 0 & 0 \\ \alpha_1 & r & 0 & 0 \\ \alpha_2 & 0 & -(a + v^* + \delta) & (N - z^*) \\ \alpha_3 & 0 & \alpha_4 & \alpha_5 \end{array} \right],$$

where $\alpha_1, \alpha_2, \alpha_3$ are the values corresponding to the above system (which are irrelevant for computation of the eigenvalues), and α_4 and α_5 are given by $L^*[2c(N - z^*) - rV'(v^*)]$ and $r + \delta N/(N - z^*)$, respectively. The eigenvalues of the system, denoted by Δ_i can now be easily calculated to be as follows: $\Delta_1 = -k$, $\Delta_2 = r$, Δ_3 and Δ_4 are the solutions of the following quadratic equation:

$$\Delta^2 - r\Delta + M^* = 0,$$

where M^* is defined by

$$M^* = -(a + v^* + \delta)(r + a + v^* + \delta) - V'(v^*)(r + 2(a + v^* + \delta))/V''(v^*),$$

where in the last two equations, z^* was substituted from eq. (A.1) of appendix 1, and the constant c was substituted from eq. (A.2') of appendix 1.

Since $M^* < 0$, the quadratic equation in Δ yields two real solutions, one of which is negative and the other positive. Altogether we have four distinct roots, two positive and two negative and thus this equilibrium is conditionally stable, i.e., there exists a real two-dimensional manifold S , containing the equilibrium point such that any solution starting on the manifold will converge to the equilibrium. Uniqueness is guaranteed in the same way as in the two-dimensional saddle point, i.e., any solution not on the manifold will not converge to the equilibrium point.

Appendix 3: On pulsing policies

The objective of this appendix is to show that the conditions under which Naslund (1979) found pulsing to be optimal are rarely met.

The condition is that the discriminant $(\alpha a - \beta b)^2 + 4bma$ is negative. For this to happen, one or all of the parameters a , b and m are negative, and

$$|4bma| > (\alpha a - \beta b)^2.$$

Since b cannot be negative [see Nicosia (1966, p.240) on whose model Naslund is basing his analysis], then the negativity of bma implies the negativity of either a or m , but not both. The parameter m is defined as the parameter converting attitude into motivation, where attitude is the feeling towards the whole product category and motivation is the stronger feeling towards the specific brand.

Thus, if m is negative, the more favorable the consumer feels about the whole product category, the less favorable he will be towards the brand in question. This is indeed a rather peculiar situation. Furthermore, for a new product, we can check the sign of the sales (X_1 in Naslund notations). Initially, at t_0 sales of the new product are at zero, i.e., $X_1(t_0) = 0$. The optimal policy is to start advertising at capacity and thus, checking the sign of the derivative of X_2 at t_0 , it is clear that at t_0^+ , $X_2(t_0^+) > 0$, thus dX_1/dt at t_0^+ is negative. Thus $X_1(t_0^+) < 0$.

This contradicts the requirement that $X_1(t) > 0$ for all t .

Thus the parameter m cannot be negative and it is the parameter a which is negative. Nicosia does not discuss this possibility at all when dealing with the underdamped (negative discriminant) case apparently because of its lack of intuitive appeal. Negativity of a implies that positive changes in motivation will reduce sales. Suppose we do accept this possibility and check whether the discriminant can be negative under some plausible conditions. Nicosia's model was compared to a model similar to this one by Dodson and Muller (1978). The latter model differs from the one given here by the fact that the trial advertising v does not appear there but a forgetting parameter ϕ was added. From the comparison, it is evident that $|4bma| = -a\delta$ (since one of them, by assumption, is negative) and $(\alpha a - \beta b)^2 = (a - \delta)^2$. Thus,

$$(\alpha a - \beta b)^2 + 4bma = (a - \delta)^2 + a\delta = a^2 - a\delta + \delta^2.$$

This is always positive since $-a\delta > 0$. Thus the discriminant is not negative even if a is.

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